PARTIAL DIFFERENTIAL EQUATIONS

BAMAT-202

Self Learning Material

Directorate of Distance Education

SWAMI VIVEKANAND SUBHARTI UNIVERSITY

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UNIT IV

Course II

Course Name: Partial Differential Equations (PDE) **Course Code:** BAMAT-202

Course Objectives:	The main objectives of this course are to introduce the students to the exciting world of partial Differential Equations and their applications.
Unit 1:	Definition of partial differential equations, order and degree of partial differential equations, Lagrange solution of linear partial differential equations of first order, working rule to find the solution of Lagrange equation Non linear PDE of first order: Charpit's method
Unit 2:	Linear partial differential equation of second and higher order of homogeneous and non homogeneous forms with constant coefficients, solution of a liner partial differential equations with constant coefficients, Determination of C.F. and the P.I
Unit 3:	partial differential equation of second order, Second order PDE with variable coefficients. Canonical Forms, Monge's method. Monge's method of integrating.
Unit 4:	Solution of heat and wave equations in one dimensions by method of separation of variables. Solution of heat and wave equations in two dimensions by method of separation of variables.

Course Learning Outcomes: The course will enable the students to:

1. Formulate Partial Differential Equations for various Mathematical models.
2. Will be able to solve partial differential equation of first and higher order using various techniques.
3. Apply these techniques to solve and analyze various mathematical models.

References:

1. Edwards, C. Henry, Penney, David E., & Calvis, David T. (2015). *Differential Equation and Boundary Value Problems*: *Computing and Modeling* (5th ed.). Pearson Education.
2. Ross, Shepley L. (2004). *Differential Equations* (3rd ed.). John Wiley & Sons. India

NOTES

UNIT I

1. PARTIAL DIFFERENTIAL EQUATIONS

STRUCTURE

INTRODUCTION

Partial differential equations arise in applied mathematics and mathematical physics when the functions involved depend on two or more independent variables. The use of partial differential equation is enormous as compared to that of ordinary differential equations. In the present chapter, we shall learn the method of solving various types of partial differential equations.

DEFINITION OF A PARTIAL DIFFERENTIAL EQUATION

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is called a **partial differential equation.**

The following are some of the examples of partial differential equations :

1. $\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}=5z+\tan(y-3x)$

2. $xz\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}=xy$

3. $(y^2+z^2)\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=-xz$

4. $x\frac{\partial z}{\partial x}+3y\frac{\partial z}{\partial y}=2\left(z-x^2\left(\frac{\partial z}{\partial y}\right)^2\right)$

5. $2\frac{\partial^2 z}{\partial x^2}-3\frac{\partial^2 z}{\partial x\partial y}-2\frac{\partial^2 z}{\partial y^2}=0$

6. $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}-y\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial x}=0.$

NOTES

ORDER OF A PARTIAL DIFFERENTIAL EQUATION

The **order** of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation. For the partial differential equations (1–6) given above, the order of the first four equations are one each and the order of the last two equations are two each.

LINEAR PARTIAL DIFFERENTIAL EQUATION

A partial differential equation is said to be **linear** if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied together. A partial differential equation which is not linear is called **non-linear.** Out of partial differential equations (1–6) given above the first, fifth and sixth equations are linear and others are non-linear.

The partial differential equation $z \frac{\partial z}{\partial x} + 5y = 7$ is not a linear partial differential equation because the dependent variable z and its partial derivative $\frac{\partial z}{\partial x}$ are multiplied together.

NOTATION

If $z = f(x, y)$ be a function of two independent variables x and y, then we shall use the following notation :

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

FORMATION OF A PARTIAL DIFFERENTIAL EQUATION

There are two ways of forming partial differential equations depending on the given relation between variables. A relation between variables may contain arbitrary constants and arbitrary functions. The elimination of arbitrary constants (or functions) give rise to a partial differential equation.

FORMATION OF A PARTIAL DIFFERENTIAL EQUATION BY ELIMINATION OF ARBITRARY CONSTANTS

Let z be a function of two independent variables x and y defined by

$$f(x, y, z, a, b) = 0, \qquad \text{...(1)}$$

where a and b are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0$$

$\Rightarrow$ $$\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \quad ...(2)$$

and $$\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \quad ...(3)$$

In general, a and b may be eliminated from (1), (2), (3) and we get an equation of the form $g(x, y, z, p, q) = 0$.

This is the required partial differential equation. The order of this equation shall be one.

Remark 1. If the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants shall usually give rise to more than one differential equation of order one. For example, if $z = \lambda x + y$, then we have differential equations $p=\frac{z-y}{x}$ and $q = 1$.

2. If the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants shall give rise to a partial differential equation of order usually greater than one.

SOLVED EXAMPLES

Example 1. *Form partial differential equations by eliminating arbitrary constants from the following relations :*

(*i*) $z = (x + a)(y + b)$ (*ii*) $z = ax^2 + by^2 + ab$

(*iii*) $z = (x^2 + a)(y^2 + b)$ (*iv*) $z = ae^{bx} \sin by$.

Sol. (*i*) We have $z = (x + a)(y + b)$ ...(1)

$\Rightarrow$ $z = xy + ay + bx + ab$

Differentiating z partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = y(1) + 0 + b(1) + 0 \quad ...(2)$$

and $$\frac{\partial z}{\partial y} = x(1) + a(1) + 0 + 0 \quad ...(3)$$

(2) $\Rightarrow$ $b=\frac{\partial z}{\partial x}-y$ (3) $\Rightarrow$ $a=\frac{\partial z}{\partial y}-x$

Putting the values of a and b in (1), we get

$$z=\left(x+\frac{\partial z}{\partial y}-x\right)\left(y+\frac{\partial z}{\partial x}-y\right) \text{ or } \mathbf{z}=\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$$

(*ii*) We have $z = ax^2 + by^2 + ab$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x}=2ax+0+0 \quad ...(2) \quad \text{and} \quad \frac{\partial z}{\partial y}=0+2by+0 \quad ...(3)$$

(2) $\Rightarrow$ $p = 2ax \Rightarrow a=\frac{p}{2x}$ (3) $\Rightarrow$ $q = 2by \Rightarrow b=\frac{q}{2y}$.

Putting the values of a and b in (1), we get

$$z = \left(\frac{p}{2x}\right)x^2 + \left(\frac{q}{2y}\right)y^2 + \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right) \quad \text{or} \quad z = \frac{px}{2} + \frac{qy}{2} + \frac{pq}{4xy}$$

or $$\mathbf{4xyz = 2px^2y + 2qxy^2 + pq.}$$

(*iii*) We have $\quad z = (x^2 + a)(y^2 + b)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = (y^2 + b)(2x + 0) \quad ...(2) \quad \text{and} \quad \frac{\partial z}{\partial y} = (x^2 + a)(2y + 0)$$ ...(3)

(2) $\Rightarrow$ $p = 2x(y^2 + b)$ $\Rightarrow$ $y^2 + b = \dfrac{p}{2x}$

(3) $\Rightarrow$ $q = 2y(x^2 + a)$ $\Rightarrow$ $x^2 + a = \dfrac{q}{2y}$

Putting the values of $y^2 + b$ and $x^2 + a$ in (1), we get

$$z = \frac{q}{2y} \cdot \frac{p}{2x} \quad \text{or} \quad \mathbf{pq = 4xyz.}$$

(*iv*) We have $\quad z = ae^{bx} \sin by$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = (a \sin by) \,.\, be^{bx}$$ ...(2)

and $$\frac{\partial z}{\partial y} = (ae^{bx}) \,.\, b \cos by$$ ...(3)

(2) $\Rightarrow$ $p = abe^{bx} \sin by$ ...(4)

(3) $\Rightarrow$ $q = abe^{bx} \cos by$ ...(5)

Dividing (4) by (5), we get $\dfrac{p}{q} = \tan by$.

Also, (4) $\Rightarrow$ $p = bz$ $\Rightarrow$ $b = \dfrac{p}{z}$.

$$\therefore \quad \frac{p}{q} = \tan\left(\frac{p}{z}\right)y \quad \text{or} \quad \mathbf{p = q \tan \frac{py}{z}}.$$

Example 2. *Find a partial differential equation by eliminating a, b and c from*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Sol. We have $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$ ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{2x}{a^2} + 0 + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2x + a^2z\frac{\partial z}{\partial x} = 0$$ ...(2)

and $$0 + \frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2y + b^2z\frac{\partial z}{\partial y} = 0$$ ...(3)

Differentiating (2) partially w.r.t. y, we get

$$0 + a^2\left(z\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\right) = 0 \quad \Rightarrow \quad \mathbf{z\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0.}$$

Example 3. *Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.*

Sol. Let $lx + my + nz = a$ ...(1)

be the equation of a plane where l, m, n are d.c.'s of the normal to the plane.

Differentiating (1) partially w.r.t. x and y, we get

$$l(1) + 0 + n\frac{\partial z}{\partial x} = 0 \quad ...(2) \qquad \text{and} \qquad 0 + m(1) + n\frac{\partial z}{\partial y} = 0 \quad ...(3)$$

(2) $\Rightarrow$ $l + np = 0$ or $l = -np$

(3) $\Rightarrow$ $m + nq = 0$ or $m = -nq$

Also $l^2 + m^2 + n^2 = 1$

$\therefore$ $(-np)^2 + (-nq)^2 + n^2 = 1$

or $(p^2 + q^2 + 1)\, n^2 = 1$ or $n = \dfrac{1}{\sqrt{p^2+q^2+1}}$ (Assuming $n > 0$)

$$\therefore \quad l = -np = -\frac{p}{\sqrt{p^2+q^2+1}}, \quad m = -nq = -\frac{q}{\sqrt{p^2+q^2+1}}$$

Putting the values of l, m and n in (1), we get

$$-\frac{px}{\sqrt{p^2+q^2+1}} - \frac{qy}{\sqrt{p^2+q^2+1}} + \frac{z}{\sqrt{p^2+q^2+1}} = a$$

or $$\mathbf{z = px + qy + a\sqrt{p^2+q^2+1}.}$$

Example 4. *Find the differential equation of the family of spheres of radius 7 with centres on the plane $x - y = 0$.*

Sol. Let (a, a, b) be any point on the plane $x - y = 0$.

$\therefore$ With centre at (a, a, b), the equation of the sphere of radius 7 is

$(x - a)^2 + (y - a)^2 + (z - b)^2 = 49$ ...(1)

$\therefore$ (1) represents a family of spheres where a and b are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$2(x - a) + 0 + 2(z - b)\, p = 0$ ...(2)

and $0 + 2(y - a) + 2(z - b)\, q = 0$ ...(3)

(2) $\Rightarrow$ $x - a = -(z - b)p$ and (3) $\Rightarrow$ $y - a = -(z - b)q$

$\therefore$ (1) $\Rightarrow$ $(z - b)^2p^2 + (z - b)^2q^2 + (z - b)^2 = 49$

$\Rightarrow$ $(p^2 + q^2 + 1)\,(z - b)^2 = 49$ ...(4)

(2) – (3) $\Rightarrow$ $2(x - y) = -2(z - b)\,(p - q)$ $\Rightarrow$ $z - b = -\dfrac{x-y}{p-q}$

$$\therefore \quad (4) \Rightarrow (p^2+q^2+1)\left(\frac{x-y}{p-q}\right)^2 = 49$$

or $$\mathbf{(p^2 + q^2 + 1)(x - y)^2 = 49(p - q)^2.}$$

NOTES

Example 5. *Show that the differential equation of all cones which have their vertex at the origin is* $px + qy = z$.

Sol. The equation of the family of cones is the homogeneous equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad ...(1)$$

where a, b, c, f, g, h are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$$2ax + 0 + 2czp + 2fyp + 2g(z.1 + xp) + 2hy = 0 \quad ...(2)$$

and $$0 + 2by + 2czq + 2f(yq + z.1) + 2gxq + 2hx = 0 \quad ...(3)$$

(2) $\Rightarrow$ $$ax + gz + hy + p(cz + fy + gx) = 0 \quad ...(4)$$

(3) $\Rightarrow$ $$by + fz + hx + q(cz + fy + gx) = 0 \quad ...(5)$$

Multiplying (4) by x and (5) by y and adding, we get

$$ax^2 + gxz + hxy + by^2 + fyz + hxy + (px + qy)(cz + fy + gx) = 0$$

or $$ax^2 + by^2 + fyz + gzx + 2hxy + (px + qy)(cz + fy + gx) = 0$$

Using (1), we get $$-(cz^2 + fyz + gzx) + (px + qy)(cz + fy + gx) = 0$$

$\Rightarrow$ $$(cz + fy + gx)(-z + px + qy) = 0 \quad \Rightarrow \quad \mathbf{px + qy = z.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. For a given relation involving variables and arbitrary constants, the relation is differentiated partially w.r.t. independent variables and arbitrary constants are eliminated to get the corresponding partial differential equation.

Rule II. If the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants shall usually give rise to more than one differential equation of order one.

Rule III. If the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to one differential equation of order one.

Rule IV. If the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants shall give rise to a differential equation of order usually greater than one.

EXERCISE A

Form partial differential equations by eliminating arbitrary constants from the following relations (Q. no. 1–10) :

1. $az + b = a^2x + y$

2. $z = ax + (1 - a)y + b$

3. $z = ax + by + ab$

4. $z = ax + a^2y^2 + b$

5. $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

6. $z = axe^y + \dfrac{1}{2}a^2e^{2y} + b$

7. $z = xy + y\sqrt{x^2 - a^2} + b$

8. $ax^2 + by^2 + cz^2 = 1$

9. $z = ax^2 + bxy + cy^2$

10. $z = ax + by + cxy$.

11. Form a partial differential equation by eliminating a and b from the equation

$$(x - a)^2 + (y - b)^2 + z^2 = k^2.$$

12. Find the partial differential equation of planes having equal x and y intercepts.

13. Find the differential equation of all spheres of fixed radius and having their centres in the xy-plane.

14. Find the differential equation of all spheres whose centre lies on z-axis.

NOTES

Answers

1. $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 1$ **2.** $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$ **3.** $z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$

4. $\frac{\partial z}{\partial y} = 2y\left(\frac{\partial z}{\partial x}\right)^2$ **5.** $2z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ **6.** $\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$

7. $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ **8.** $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial x \partial y} = 0$ **9.** $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

10. $\frac{\partial^2 z}{\partial x^2} = 0, \frac{\partial^2 z}{\partial y^2} = 0, z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y}$ **11.** $z^2\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right) = k^2$

12. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ **13.** $z^2\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right) = k^2$ **14.** $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$.

FORMATION OF A PARTIAL DIFFERENTIAL EQUATION BY ELIMINATION OF ARBITRARY FUNCTIONS

Let u and v be independent functions of three variables x, y, z and let

$$f(u, v) = 0 \quad \text{...(1)}$$

be an arbitrary relation between u and v. Regarding z as a function of x, y and differentiating (1) partially w.r.t. x, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) \quad \text{...(2)}$$

Similarly, differentiating (1) partially w.r.t. y, we get

$$\frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \quad \text{...(3)}$$

Eliminating f using (2) and (3), we get

$$-\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \mathrm{P}p + \mathrm{Q}q = \mathrm{R},$$

where $\mathrm{P} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$, $\mathrm{Q} = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$ and $\mathrm{R} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

This is the required partial differential equation. The order of this equation is one.

Remark. The functions u and v are said to be *independent* if u/v is not merely a constant.

NOTES

SOLVED EXAMPLES

Example 6. *Form partial differential equations by eliminating arbitrary function from the following relations :*

(*i*) $z = f(x^2 + 2y^2)$ (*ii*) $f(x^2 + y^2 + z^2) = ax + by + cz$

(*iii*) $f(x^2 + y^2, z - xy) = 0$ (*iv*) $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0.$

Sol. (*i*) We have $z = f(x^2 + 2y^2)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 + 2y^2)\frac{\partial}{\partial x}(x^2 + 2y^2) = 2x\, f'(x^2 + 2y^2)$$

and
$$\frac{\partial z}{\partial y} = q = f'(x^2 + 2y^2)\frac{\partial}{\partial y}(x^2 + 2y^2) = 4y\, f'(x^2 + 2y^2)$$

Dividing we get $\frac{p}{q} = \frac{x}{2y}$ or $\mathbf{2py - qx = 0.}$

(*ii*) We have $f(x^2 + y^2 + z^2) = ax + by + cz.$ ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$$f'(x^2 + y^2 + z^2).\left(2x + 0 + 2z\frac{\partial z}{\partial x}\right) = a.1 + 0 + c\frac{\partial z}{\partial x} \quad ...(2)$$

and
$$f'(x^2 + y^2 + z^2).\left(0 + 2y + 2z\frac{\partial z}{\partial y}\right) = 0 + b.1 + c\frac{\partial z}{\partial y} \quad ...(3)$$

(2) $\Rightarrow$ $$2f'(x^2 + y^2 + z^2).(x + pz) = a + cp \quad ...(4)$$

(3) $\Rightarrow$ $$2f'(x^2 + y^2 + z^2).(y + qz) = b + cq \quad ...(5)$$

Dividing (4) by (5), we get

$$\frac{x + pz}{y + qz} = \frac{a + cp}{b + cq} \text{ or } (x + pz)(b + cq) = (y + qz)(a + cp)$$

or $$bx + cxq + bpz + czpq = ay + cyp + aqz + czpq$$

or $$\mathbf{(bz - cy)p + (cx - az)q = ay - bx.}$$

(*iii*) We have $f(x^2 + y^2, z - xy) = 0.$ ...(1)

Let $u = x^2 + y^2$ and $v = z - xy.$

$\therefore$ (1) $\Rightarrow$ $f(u, v) = 0$ ...(2)

Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}.2x + \frac{\partial f}{\partial v}(-y + 1.p) = 0$$

$$\Rightarrow \quad 2x\frac{\partial f}{\partial u} + (p - y)\frac{\partial f}{\partial v} = 0 \quad ...(3)$$

Differentiating (2) partially w.r.t. y, we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}.2y + \frac{\partial f}{\partial v}(-x + 1.q) = 0$$

$\Rightarrow \qquad 2y\dfrac{\partial f}{\partial u} + (q - x)\dfrac{\partial f}{\partial v} = 0$ ...(4)

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (3) and (4), we get $\begin{vmatrix} 2x & p - y \\ 2y & q - x \end{vmatrix}^* = 0$

$\Rightarrow \qquad 2x(q - x) - 2y(p - y) = 0$ or $\mathbf{xq - yp = x^2 - y^2}$.

(*iv*) We have $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$.

$\therefore$ (1) $\Rightarrow \qquad f(u, v) = 0$ ...(2)

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

$\Rightarrow \qquad \dfrac{\partial f}{\partial u}(2x + 2zp) + \dfrac{\partial f}{\partial v}(-2y + 2zp) = 0$

$\Rightarrow \qquad (x + zp)\dfrac{\partial f}{\partial u} + (zp - y)\dfrac{\partial f}{\partial v} = 0$ ...(3)

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0$$

$\Rightarrow \qquad \dfrac{\partial f}{\partial u}(2y + 2zq) + \dfrac{\partial f}{\partial v}(-2x + 2zq) = 0$

$\Rightarrow \qquad (y + zq)\dfrac{\partial f}{\partial u} + (zq - x)\dfrac{\partial f}{\partial v} = 0$ ...(4)

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (3) and (4), we get $\begin{vmatrix} x + zp & zp - y \\ y + zq & zq - x \end{vmatrix} = 0.$

$\Rightarrow \qquad (x + zp)(zq - x) - (y + zq)(zp - y) = 0$

$\Rightarrow \qquad xzq - x^2 + z^2pq - xzp - yzp + y^2 - z^2pq + yzq = 0$

$\Rightarrow \qquad (x + y)zp - (x + y)zq = y^2 - x^2 \quad \Rightarrow \quad \mathbf{z(p - q) = y - x}.$

Example 7. *Form partial differential equations by eliminating arbitrary functions from the following relations :*

(*i*) $z = x\phi(y) + y\psi(x)$ (*ii*) $z = f(x^2 - y) + g(x^2 + y)$

(*iii*) $x = f(z) + g(y)$ (*iv*) $z = f(y + ax) + g(y + bx),\ a \neq b$.

Sol. (*i*) We have $z = x\phi(y) + y\psi(x)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$\dfrac{\partial z}{\partial x} = \phi(y).1 + y\psi'(x)$ ...(2) and $\dfrac{\partial z}{\partial y} = x\phi'(y) + \psi(x).1$ ...(3)

Differentiating (2) w.r.t. y, we get

$\dfrac{\partial^2 z}{\partial y \partial x} = \phi'(y) + \psi'(x).1$ ...(4)

***Why this step.** If $ax + by = 0$ and $cx + dy = 0$, then $\dfrac{x}{y} = -\dfrac{b}{a}$ and $\dfrac{x}{y} = -\dfrac{d}{c}$.

$\therefore$ Eliminating x, y, we get $-\dfrac{b}{a} = -\dfrac{d}{c}$ or $ad - bc = 0$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$

NOTES

$$\Rightarrow \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\left(\frac{\partial z}{\partial y} - \psi(x)\right) + \frac{1}{y}\left(\frac{\partial z}{\partial x} - \phi(y)\right) \qquad \text{(Using (2) and (3))}$$

$$\Rightarrow \qquad xy\,\frac{\partial^2 z}{\partial y \partial x} = x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} - (x\phi(y) + y\psi(x))$$

$$\Rightarrow \qquad \mathbf{xy\,\frac{\partial^2 z}{\partial y \partial x} = x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} - z}.$$

(*ii*) We have $z = f(x^2 - y) + g(x^2 + y)$. ...(1)

Differentiating partially w.r.t. x and y, we get

$$p = f'(x^2 - y)\,\frac{\partial}{\partial x}(x^2 - y) + g'(x^2 + y)\,\frac{\partial}{\partial x}(x^2 + y) \qquad ...(2)$$

and $$q = f'(x^2 - y)\,\frac{\partial}{\partial y}(x^2 - y) + g'(x^2 + y)\,\frac{\partial}{\partial y}(x^2 + y) \qquad ...(3)$$

(2) $\Rightarrow$ $p = 2x\,f'(x^2 - y) + 2x\,g'(x^2 + y)$ ...(4)

(3) $\Rightarrow$ $q = -f'(x^2 - y) + 1\,.\,g'(x^2 + y)$ ...(5)

Differentiating (4) w.r.t. x, we get

$$r = 2x\,f''(x^2 - y)\,.\,2x + 2.1\,f'(x^2 - y) + 2x\,g''(x^2 + y)\,.\,2x + 2.1\,g'(x^2 + y)$$

or $$r = 4x^2\,(f''(x^2 - y) + g''(x^2 + y)) + 2\,(f'(x^2 - y) + g'(x^2 + y)) \qquad ...(6)$$

Differentiating (5) w.r.t. y, we get

$$t = -f''(x^2 - y)\,.\,(-1) + g''(x^2 + y)\,.\,1$$

or $t = f''(x^2 - y) + g''(x^2 + y)$ ...(7)

$\therefore$ (6) $\Rightarrow$ $r = 4x^2 t + 2\left(\dfrac{p}{2x}\right)$ (Using (4) and (7))

$$\Rightarrow \qquad \mathbf{x\,\frac{\partial^2 z}{\partial x^2} = 4x^3\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x}}.$$

(*iii*) We have $x = f(z) + g(y)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$1 = f'(z)\,p + 0$ ...(2) and $0 = f'(z)\,q + g'(y)$ ...(3)

Differentiating (2) and (3) w.r.t. x, we get

$0 = f''(z)\,p\,.\,p + f'(z)\,r$ ...(4) and $0 = f''(z)\,p\,.\,q + f'(z)\,s + 0$ ...(5)

(4) $\Rightarrow$ $f''(z)\,p^2 = -f'(z)\,r$

(5) $\Rightarrow$ $f''(z)\,pq = -f'(z)\,s$

Dividing, we get $\dfrac{p}{q} = \dfrac{r}{s}$ or $\mathbf{ps - qr = 0.}$

(*iv*) We have $z = f(y + ax) + g(y + bx)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$p = f'(y + ax)\,.\,a + g'(y + bx)\,.\,b$ ...(2)

and $q = f'(y + ax)\,.\,1 + g'(y + bx)\,.\,1$ ...(3)

Differentiating (2) partially w.r.t. x and y, we get

$r = f''(y + ax)\,a^2 + g''(y + bx)\,b^2$ ...(4)

and $s = f''(y + ax)\,a\,.\,1 + g''(y + bx)\,b\,.\,1$ ...(5)

Differentiating (3) w.r.t. y, we get

$$t = f''(y + ax) \,.\, 1 + g''(y + bx) \,.\, 1 \qquad ...(6)$$

(4) $\Rightarrow$ $a^2 f''(y + ax) + b^2 g''(y + bx) - r = 0$

(5) $\Rightarrow$ $a f''(y + ax) + b\, g''(y + bx) - s = 0$

(6) $\Rightarrow$ $f''(y + ax) + g''(y + bx) - t = 0$

Eliminating $f''(y + ax)$ and $g''(y + bx)$ from these three equations, we get

$$\begin{vmatrix} a^2 & b^2 & r \\ a & b & s \\ 1 & 1 & t \end{vmatrix} = 0$$

$\Rightarrow$ $(a - b)\, r - (a^2 - b^2)\, s + (a^2b - ab^2)\, t = 0$

$\Rightarrow$ $\mathbf{r - (a + b)s + abt = 0.}$

Example 8. *The equation of any cone with vertex at* $P(x_0, y_0, z_0)$ *is of the form*

$$f\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0.$$

Find the differential equation.

Sol. We have
$$f\left(\frac{x - x_0}{z - z_0}, \frac{y - y_0}{z - z_0}\right) = 0. \qquad ...(1)$$

Let
$$u = \frac{x - x_0}{z - z_0} \quad \text{and} \quad v = \frac{y - y_0}{z - z_0}.$$

$\therefore$ (1) $\Rightarrow$ $f(u, v) = 0$ ...(2)

Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial f}{\partial u}\left(\frac{1 - 0}{z - z_0} - \frac{x - x_0}{(z - z_0)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y - y_0}{(z - z_0)^2}\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}\left(\frac{1}{z - z_0} - p\frac{x - x_0}{(z - z_0)^2}\right) + \frac{\partial f}{\partial v}\left(-p\frac{y - y_0}{(z - z_0)^2}\right) = 0 \qquad ...(3)$$

Differentiating (2) partially w.r.t. y, we get

$$\frac{\partial f}{\partial u}\left(-\frac{x - x_0}{(z - z_0)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1 - 0}{z - z_0} - \frac{y - y_0}{(z - z_0)^2}\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}\left(-q\frac{x - x_0}{(z - z_0)^2}\right) + \frac{\partial f}{\partial v}\left(\frac{1}{z - z_0} - q\frac{y - y_0}{(z - z_0)^2}\right) = 0 \qquad ...(4)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4), we get

$$\begin{vmatrix} \frac{1}{z - z_0} - p\frac{x - x_0}{(z - z_0)^2} & -p\frac{y - y_0}{(z - z_0)^2} \\ -q\frac{x - x_0}{(z - z_0)^2} & \frac{1}{z - z_0} - q\frac{y - y_0}{(z - z_0)^2} \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} z - z_0 - p(x - x_0) & -p(y - y_0) \\ -q(x - x_0) & z - z_0 - q(y - y_0) \end{vmatrix} = 0$$

$$\Rightarrow \quad [z - z_0 - p(x - x_0)]\,[z - z_0 - q(y - y_0)] - pq(x - x_0)(y - y_0) = 0$$

$$\Rightarrow \quad (z - z_0)^2 - p(x - x_0)(z - z_0) - (z - z_0)\, q(y - y_0) = 0$$

$$\Rightarrow \quad \mathbf{p(x - x_0) + q(y - y_0) = z - z_0.}$$

NOTES

WORKING RULES FOR SOLVING PROBLEMS

Rule I. For a given relation involving variables and arbitrary functions, the relation is differentiated partially w.r.t. independent variables and arbitrary functions are eliminated to get the corresponding partial differential equation.

Rule II. If the number of arbitrary functions is less than the number of independent variables, then the elimination of arbitrary functions shall give rise to a differential equation of order one.

Rule III. If the number of arbitrary functions is equal to the number of independent variables, then the elimination of arbitrary functions shall give rise to a differential equation of order usually greater than one.

EXERCISE B

Find partial differential equation by eliminating arbitrary functions from the following relations :

1. $z = f(x + ky)$

2. $z = f(x^2 - y^2)$

3. $f(x^2 + y^2 + z^2) = x + y + z$

4. $z = x + y + f(xy)$

5. $z = xy + f(x^2 + y^2)$

6. $z = f(xy/z)$

7. $f(x + y + z) = xyz$

8. $z = (x + y)\, f(x^2 - y^2)$

9. $z = f(x) + e^y\, g(x)$

10. $z = f(xy) + g(x + y)$

11. $z = f(xy) + g(x/y)$

12. $f(x + y + z,\, x^2 + y^2 - z^2) = 0$

13. $z = f(x \cos \alpha + y \sin \alpha - at) + g(x \cos \alpha + y \sin \alpha + at)$.

Answers

1. $q = kp$ **2.** $yp + xq = 0$ **3.** $(y - z)\, p + (z - x)\, q = x - y$

4. $px - qy = x - y$ **5.** $py - qx = y^2 - x^2$ **6.** $px - qy = 0$

7. $x(y - z)p + y(z - x)q = z(x - y)$ **8.** $yp + xq = z$

9. $t - q = 0$ **10.** $x(y - x)\, r - (y^2 - x^2)\, s + y(y - x)t + (p - q)\,(x + y) = 0$

11. $x^2r - y^2t + xp - yq = 0$ **12.** $p(y + z) - (x + z)q = x - y$

13. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial t^2}$.

Hint

12. $u = x + y + z,\ v = x^2 + y^2 - z^2 \quad \Rightarrow \quad f(u, v) = 0$

Diff. w.r.t. x, we get $\dfrac{\partial f}{\partial u}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial x}\right) + \dfrac{\partial f}{\partial v}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}\dfrac{\partial z}{\partial x}\right) = 0$

$\Rightarrow \qquad \dfrac{\partial f}{\partial u}(1 + 1 \,.\, p) + \dfrac{\partial f}{\partial v}(2x - 2zp) = 0$

Similarly, $\qquad \dfrac{\partial f}{\partial u}(1 + 1 \,.\, q) + \dfrac{\partial f}{\partial v}(2y - 2zq) = 0$

Eliminating $\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}$, we get $\begin{vmatrix} 1 + p & 2x - 2zp \\ 1 + q & 2y - 2zq \end{vmatrix} = 0$.

2. PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER (Equations Linear in *p* and *q*)

STRUCTURE

INTRODUCTION

In the last chapter, we studied the methods of forming partial differential equations. The next step is to solve partial differential equations. Solving a partial differential equation means to find a function which satisfies the given partial differential equation. A function satisfying a partial differential equation is called its solution (*or integral*). In the present chapter, we shall confine ourselves to the solution of partial differential equations of first order and at the same time linear in p and q.

SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION

A **solution** of a partial differential equation is a relation between the variables by means of which the partial derivatives are derived there from the given partial differential equation is satisfied.

A solution of a partial differential equation is also called an **integral** of the equation.

In the context of partial differential equations of first order, there are four types of solutions. These are :

(*i*) Complete solution (*ii*) Particular solution
(*iii*) Singular solution (*iv*) General solution.

NOTES

COMPLETE SOLUTION

Let z be a function of two independent variables x and y defined by

$$f(x, y, z, a, b) = 0 \quad ...(1)$$

where a and b are arbitrary constants.

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \quad ...(2) \qquad \text{and} \qquad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \quad ...(3)$$

Eliminating a and b from (1), (2) and (3), we get an equation of the form $g(x, y, z, p, q) = 0$. This is a partial differential equation of first order.

In (1), the number of arbitrary constants is two which is equal to the number of independent variables in $g(x, y, z, p, q) = 0$.

The function $f(x, y, z, a, b) = 0$ is called the **complete solution** of the equation $g(x, y, z, p, q) = 0$.

For example $z = (x + a)(y + b)$ is a complete solution of the equation $pq = z$.

PARTICULAR SOLUTION

A solution obtained by giving some particular values to the arbitrary constants in the complete solution of a partial differential equation of first order is called a **particular solution** of the concerned equation.

For example $z = (x + 1)(y + 4)$ is a particular solution of the equation $pq = z$.

SINGULAR SOLUTION

Let $f(x, y, z, a, b) = 0$ be the complete solution of a partial differential equation $g(x, y, z, p, q) = 0$. The relation between x, y and z obtained by eliminating the arbitrary constants a and b between the equations $f(x, y, z, a, b) = 0, \frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0$ is called the **singular solution** of the equation $g(x, y, z, p, q) = 0$, provided it satisfies this equation. This solution represents the envelope of the surfaces represented by the complete solution of the given partial differential equation. The singular solution may or may not be contained in the complete solution of the equation.

For example, $z = ax + by - (a^2 + b^2)$ is the complete solution of the partial differential equation $z = px + qy - (p^2 + q^2)$.

Let $\quad f(x, y, z, a, b) = z - ax - by + a^2 + b^2$.

$$\therefore \quad \frac{\partial f}{\partial a} = -x + 2a, \quad \frac{\partial f}{\partial b} = -y + 2b$$

Eliminating a and b from the equations,

$$z - ax - by + a^2 + b^2 = 0, \quad -x + 2a = 0, \quad -y + 2b = 0, \text{ we get } z = \frac{x^2 + y^2}{4}.$$

This also satisfy the given equation.

$\therefore \quad z = \frac{x^2 + y^2}{4}$ is the singular solution of the equation $z = px + qy - (p^2 + q^2)$.

GENERAL SOLUTION

Let $f(x, y, z, a, b) = 0$ be the complete solution of a partial differential equation $g(x, y, z, p, q) = 0$. Let $b = \phi(a)$.

$\therefore$ $f(x, y, z, a, \phi(a)) = 0$ is a one-parameter family of the surfaces of $g(x, y, z, p, q) = 0$. The relation between x, y and z obtained by eliminating the arbitrary constant a between the equations $f(x, y, z, a, \phi(a)) = 0$ and $\frac{\partial f}{\partial a} = 0$ is called the **general solution** of the equation $g(x, y, z, p, q) = 0$, provided it satisfies this equation. This solution represents the envelope of the surfaces represented by the equation $f(x, y, z, a, \phi(a)) = 0$.

If $b = \phi(a)$, where ϕ is an arbitrary function, then the elimination of a between the equations $f(x, y, z, a, \phi(a)) = 0$, and $\frac{\partial f}{\partial a} = 0$ is not possible. Thus the general solution of the equation $g(x, y, z, p, q) = 0$ is written as the set of equations $f(x, y, z, a, \phi(a)) = 0, \frac{\partial f}{\partial a} = 0$, where ϕ is any arbitrary function.

We know that if u and v be independent functions of x, y, z and

$$f(u, v) = 0 \qquad \text{...(1)}$$

be an arbitrary function of u and v, then $\quad \mathrm{P}p + \mathrm{Q}q = \mathrm{R}$ ...(2)

where $\quad \mathrm{P} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}, \quad \mathrm{Q} = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}, \quad \mathrm{R} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$

(2) is a partial differential equation of first order.

Thus $f(u, v) = 0$ is a solution of the equation $\mathrm{P}p + \mathrm{Q}q = \mathrm{R}$. Since $f(u, v) = 0$ contains an arbitrary function 'f', it is the general solution of the equation (2).

LAGRANGE LINEAR EQUATION

We know that a partial differential equation of first order involves only the first order partial derivatives of the dependent variable (z) w.r.t. the independent variables (x and y). Thus an equation of first order involves x, y, z, p, q and may also involve powers of partial derivatives p and q.

In particular, a partial differential equation of first order and at the same time linear in p and q is of the form $\mathrm{P}p + \mathrm{Q}q = \mathrm{R}$ where P, Q, R are functions of x, y, z. This type of a partial differential equation is called a **Lagrange linear equation.**

For example $4xp + 6y^2q = x^2 + y^2 + z^2$ is a Lagrange linear equation.

SOLUTION OF LAGRANGE LINEAR EQUATION

Let $\quad \mathrm{P}p + \mathrm{Q}q = \mathrm{R}$ ...(1)

be a Lagrange linear equation where P, Q, R are functions of the dependent variable z and independent variables x and y. The system of equations

$$\frac{dx}{\mathrm{P}} = \frac{dy}{\mathrm{Q}} = \frac{dz}{\mathrm{R}} \qquad \text{...(2)}$$

is called the **Lagrange system** of ordinary differential equations for the equation (1).

Let $u = C_1$ and $v = C_2$ be two independent solutions of the equations (2).

Let $f(u, v) = 0$, be an arbitrary function of u and v. ...(3)

Differentiating (3) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

and
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$, we get $\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=0$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad ...(4)$$

∴ (3) is a solution of the equation (4).

Taking differentials of $u = C_1$ and $v = C_2$, we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \qquad ...(5)$$

and
$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \qquad ...(6)$$

Since u and v are independent functions, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \qquad ...(7)$$

Using (2) and (7), we have

$$\frac{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}{P}=\frac{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}{Q}=\frac{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}{R}=\lambda \text{ (say)}$$

$$\therefore \quad \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=\lambda P,\quad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\lambda Q,\quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\lambda R.$$

∴ Putting these values in (4), we get $\lambda Pp + \lambda Qq = \lambda R$ or $Pp + Qq = R$, which is the given partial differential equation.

∴ If $u = C_1$ and $v = C_2$ be two independent solutions of the system of differential equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, then any arbitrary function $f(u, v)$ of u and v is a solution of the Lagrange linear equation $Pp + Qq = R$. The solution $f(u, v) = 0$ is the general solution of the equation $Pp + Qq = R$. In particular, for arbitrary constants a and b, the solution $u = av + b$ is a complete solution of the equation $Pp + Qq = R$.

Remark. The Lagrange system of ordinary differential equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ for the partial differential equation $Pp+Qq=R$ is also known as the **auxiliary system of equations** or simply as the **auxiliary equations** of the equation $Pp+Qq=R$.

WORKING STEPS FOR SOLVING $Pp + Qq = R$

Step I. Indentify the functions P, Q and R.

Step II. Form the system : $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

Step III. Find two independent solutions $u = C_1$ and $v = C_2$ of the system given in **step II.**

Step IV. Write $f(u, v) = 0$ and call it the general solution of the given equation.

Type I. In this type, we shall consider the solution of the equation $Pp + Qq = R$ for which the equality of two factors of the auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ gives an equation in the variables whose differentials are involved. Two independent solutions of the auxiliary equations are calculated in this manner.

SOLVED EXAMPLES

Example 1. *Find the general solution of the following Lagrange linear equations:*

(*i*) $2p + 5q = 1$ (*ii*) $y^2p + x^2q = x^2y^2z^2$

(*iii*) $\frac{y^2z}{x}p + xzq = y^2$ (*iv*) $zp = x$

(*v*) $(x^2 + 2y^2)p - xyq = xz$.

Sol. (*i*) We have $2p + 5q = 1$.

Here $P = 2, Q = 5, R = 1$

$\therefore$ Auxiliary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

i.e., $$\frac{dx}{2}=\frac{dy}{5}=\frac{dz}{1} \quad ...(1)$$

Taking the first two fractions of (1), we get $5dx - 2dy = 0$ ...(2)

Integrating (2), we get $5x - 2y = C_1$ ...(3)

Taking the last two fractions of (1), we get $dy - 5dz = 0$ ...(4)

Integrating (4), we have $y - 5z = C_2$ ...(5)

From (3) and (5), the general solution of the given equation is $\mathbf{f(5x - 2y, y - 5z) = 0}$, where f is any arbitrary function.

(*ii*) We have $y^2p + x^2q = x^2y^2z^2$.

Here $P = y^2, Q = x^2, R = x^2y^2z^2$.

$\therefore$ Auxiliary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

i.e., $$\frac{dx}{y^2}=\frac{dy}{x^2}=\frac{dz}{x^2y^2z^2} \quad ...(1)$$

NOTES

Taking the first two fractions of (1), we get

$$x^2dx - y^2dy = 0 \text{ or } 3x^2dx - 3y^2dy = 0 \qquad ...(2)$$

Integrating (2), we get $\quad x^3 - y^3 = C_1 \qquad ...(3)$

Taking the last two fractions of (1), we get $3y^2dy - 3z^{-2}dz = 0 \qquad ...(4)$

Integrating (4), we get $\quad y^3 + 3z^{-1} = C_2 \qquad ...(5)$

From (3) and (5), the general solution of the given equation is $\mathbf{f(x^3 - y^3, y^3 + 3z^{-1}) = 0}$, where *f* is any arbitrary function.

(*iii*) We have $\quad \frac{y^2z}{x}p + xzq = y^2.$

Here $\quad P = \frac{y^2z}{x}, Q = xz, R = y^2$

$\therefore$ Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

i.e.,
$$\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad ...(1)$$

Taking the first two fractions of (1), we get

$$x^2dx = y^2dy \quad \text{or} \quad 3x^2dx - 3y^2dy = 0 \qquad ...(2)$$

Integrating (2), we have $\quad x^3 - y^3 = C_1 \qquad ...(3)$

Taking the first and last fractions of (1), we get

$$xdx = zdz \quad \text{or} \quad 2xdx - 2zdz = 0 \qquad ...(4)$$

Integrating (4), we have $\quad x^2 - z^2 = C_2 \qquad ...(5)$

From (3) and (5), the general solution of the given equation is $\mathbf{f(x^3 - y^3, x^2 - z^2) = 0,}$ where *f* is any arbitrary function.

(*iv*) We have $zp = x. \quad \therefore \quad zp + 0.q = x$

Here $\quad P = z, Q = 0, R = x.$

$\therefore$ Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

i.e.,
$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{x} \qquad ...(1)$$

Second fraction of (1) implies $dy = 0$

$\therefore \quad y = C_1 \qquad ...(2)$

Taking the first and third fractions of (1), we get

$$xdx = zdz \quad \text{or} \quad 2xdx - 2zdz = 0 \qquad ...(3)$$

Integrating (3), we have $\quad x^2 - z^2 = C_2 \qquad ...(4)$

From (2) and (4), the general solution of the given equation is $\mathbf{f(y, x^2 - z^2) = 0,}$ where *f* is any arbitrary function.

(*v*) We have $\quad (x^2 + 2y^2)\, p - xyq = xz.$

Here $\quad P = x^2 + 2y^2, \quad Q = -xy, R = xz.$

$\therefore$ Auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$

i.e.,
$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \qquad ...(1)$$

Taking the first two fractions of (1), we get $\dfrac{dx}{x^2+2y^2} = \dfrac{dy}{-xy}$.

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x^2+2y^2}{-xy} \quad \Rightarrow \quad 2x\frac{dx}{dy} = -\frac{2x^2}{y} - 4y$$

$$\Rightarrow \quad 2x\frac{dx}{dy} + x^2\left(\frac{2}{y}\right) = -4y \qquad \text{...(2)}$$

Let $z = x^2$.

$$\therefore \quad (2) \Rightarrow \quad \frac{dz}{dy} + z\left(\frac{2}{y}\right) = -4y$$

This is a linear differential equation of order one.

$$\text{I.F.} = e^{\int \frac{2}{y}dy} = e^{2\log y} = y^2$$

$$\therefore \quad zy^2 = \int (-4y)\,y^2\,dy + C_1 \quad \text{or} \quad x^2y^2 + y^4 = C_1 \qquad \text{...(3)}$$

Taking the last two fractions of (1), we get $\dfrac{dy}{y} + \dfrac{dz}{z} = 0$ ...(4)

Integrating (4), we get $\log y + \log z = \log C_2$ or $yz = C_2$ ...(5)

From (3) and (5), the general solution of the given equation is $\mathbf{f(x^2y^2 + y^4, yz) = 0}$, where f is any arbitrary function.

Type II. In this type, we shall consider the solution of the equation $Pp + Qq = R$ for which the equality of two factors of the auxiliary equations $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ gives an equation in the variables whose differentials are involved. Another independent solution of the auxiliary equations is found by using the first solution.

Example 2. *Find the general solution of the following Lagrange linear equations:*

(*i*) $p + 2q = 5z + \tan(y - 2x)$ (*ii*) $yp + xq = xyz^2 (x^2 - y^2)$

(*iii*) $xz(z^2 + xy)\,p - yz(z^2 + xy)\,q = x^4$ (*iv*) $z(p - q) = z^2 + (x + y)^2$.

Sol. (*i*) We have $p + 2q = 5z + \tan(y - 2x)$.

$$\therefore \quad \text{Auxiliary equations are} \quad \frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z + \tan(y - 2x)}. \qquad \text{...(1)}$$

Taking the first two fractions of (1), we get

$$dy - 2dx = 0 \qquad \text{...(2)}$$

Integrating (2), we have $y - 2x = C_1$ ...(3)

Taking the last two fractions of (1) and using (3), we have

$$\frac{dy}{2} - \frac{dz}{5z + \tan C_1} = 0 \qquad \text{...(4)}$$

Integrating (4), we have $\dfrac{1}{2}y - \dfrac{1}{5}\log|5z + \tan C_1| = C_2$

or $$5y - 2\log|5z + \tan(y - 2x)| = 10C_2 \qquad \text{...(5)}$$

From (3) and (5), the general solution of the given equation is

$$\mathbf{f(y - 2x, 5y - 2\log|5z + \tan(y - 2x)|) = 0,}$$

where f is any arbitrary function.

(*ii*) We have $yp + xq = xyz^2 (x^2 - y^2)$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2 (x^2 - y^2)}$. ...(1)

Taking the first two fractions of (1), we get $2xdx - 2ydy = 0$ ...(2)

Integrating (2), we have $x^2 - y^2 = C_1$ ...(3)

Taking the last two fractions of (1) and using (3), we have $ydy - \dfrac{dz}{C_1 z^2} = 0$ ...(4)

Integrating (4), we have $\dfrac{y^2}{2} - \dfrac{1}{C_1}\left(\dfrac{z^{-1}}{-1}\right) = C_2$

$\Rightarrow$ $\dfrac{y^2}{2} + \dfrac{1}{z(x^2 - y^2)} = C_2$ ...(5)

From (3) and (5), the general solution of the given equation is

$$\mathbf{f\left(x^2 - y^2, \frac{y^2}{2} + \frac{1}{z(x^2 - y^2)}\right) = 0}$$, where *f* is any arbitrary function.

(*iii*) We have $xz(z^2 + xy)\, p - yz(z^2 + xy)\, q = x^4$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{xz(z^2 + xy)} = \dfrac{dy}{-yz(z^2 + xy)} = \dfrac{dz}{x^4}$. ...(1)

Taking the first two fractions of (1), we get $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$ ...(2)

Integrating (2), we have $\log x + \log y = \log C_1$ or $xy = C_1$ ...(3)

Taking the first and third fractions of (1) and using (3), we get $\dfrac{dx}{z(z^2 + C_1)} = \dfrac{dz}{x^3}$

or $x^3 dx - (z^3 + C_1 z)\, dz = 0$ ...(4)

Integrating (4), we have $\dfrac{x^4}{4} - \left(\dfrac{z^4}{4} + \dfrac{C_1 z^2}{2}\right) = C_2$

or $x^4 - z^4 - 2xyz^2 = 4C_2$ ...(5)

From (3) and (5), the general solution of the given equation is $\mathbf{f(xy, x^4 - z^4 - 2xyz^2) = 0}$, where *f* is any arbitrary function.

(*iv*) We have $z(p - q) = z^2 + (x + y)^2$.

$\Rightarrow$ $zp - zq = z^2 + (x + y)^2$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x + y)^2}$. ...(1)

Taking the first two fractions of (1), we have $dx + dy = 0$ ...(2)

Integrating, we have $x + y = C_1$ ...(3)

Taking the last two fractions of (1) and using (3), we have $dy + \dfrac{zdz}{z^2 + C_1^2} = 0$ ...(4)

Integrating (4), we get

$$y + \frac{1}{2}\log|z^2 + C_1^2| = C_2 \text{ or } y + \frac{1}{2}\log(z^2 + (x + y)^2) = C_2$$ ...(5)

NOTES

From (3) and (5), the general solution of the given equation is

$$\mathbf{f\left(x + y, y + \frac{1}{2}\log\left(z^2 + (x + y)^2\right)\right) = 0},$$

where f is any arbitrary function.

Type III. In this type, we shall consider the solution of the equation Pp + Qq = R by using the formula :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R},$$

where P_1, Q_1, R_1 are some functions of x, y and z. If for some choice of P_1, Q_1, R_1, the sum $P_1P + Q_1Q + R_1R$ is zero, then we have $P_1 dx + Q_1 dy + R_1 dz = 0$. We integrate this equation to get one solution of the auxiliary equations. P_1, Q_1, R_1 are called **multipliers.** By using different set of multipliers or by using two fractions of the auxiliary equations, we find another independent solution of the auxiliary equations.

Example 3. *Find the general solution of the following Lagrange linear equations:*

(*i*) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ (*ii*) $x(y^2 + z)\,p - y(x^2 + z)\,q = z(x^2 - y^2)$

(*iii*) $(y^2 + z^2)p - xyq + xz = 0$ (*iv*) $(4y - 3z)p + (4x - 2z)q = 2y - 3x$.

Sol. (*i*) We have $x(y^2 - z^2)\,p + y(z^2 - x^2)\,q = z(x^2 - y^2)$.

$\therefore$ Auxiliary equations are $$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \qquad ...(1)$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get $\log|x| + \log|y| + \log|z| = \log C_1$

or $$|xyz| = C_1 \quad \text{or} \quad xyz = \pm C_1 \qquad ...(2)$$

Taking x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore$ $xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, we get $x^2 + y^2 + z^2 = C_2$ ...(3)

From (2) and (3), the general solution of the given equation is $\mathbf{f(xyz, x^2 + y^2 + z^2) = 0}$, where f is any arbitrary function.

(*ii*) We have $x(y^2 + z)\,p - y(x^2 + z)\,q = z(x^2 - y^2)$.

$\therefore$ Auxiliary equations are $$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)} \qquad ...(1)$$

Taking $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + (-1)\,dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\therefore \qquad 2xdx + 2ydy - 2dz = 0$

Integrating, we get $\quad x^2 + y^2 - 2z = C_1$ ...(2)

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier, each fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\therefore \qquad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get $\log |x| + \log |y| + \log |z| = \log C_2$

or $\qquad |xyz| = C_2 \quad$ or $\quad xyz = \pm C_2$ ...(3)

From (2) and (3), the general solution of the given equation is $\mathbf{f(x^2 + y^2 - 2z, xyz) = 0}$**,** where f is any arbitrary function.

(*iii*) We have $\qquad (y^2 + z^2)\, p - xyq = -xz.$

$\therefore$ Auxiliary equations are $\quad \frac{dx}{y^2 + z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$ ...(1)

Taking the last two fractions of (1), we get $\frac{dy}{y} - \frac{dz}{z} = 0$

Integrating, we have $\log |y| - \log |z| = \log C_1$

or $\qquad \left|\frac{y}{z}\right| = C_1 \quad$ or $\quad \frac{y}{z} = \pm C_1$ ...(2)

Taking x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \qquad 2xdx + 2ydy + 2zdz = 0$

Integrating, we have $x^2 + y^2 + z^2 = C_2$ ...(3)

From (2) and (3), the general solution of the given equation is $\mathbf{f(y/z, x^2 + y^2 + z^2) = 0}$**,** where f is any arbitrary function.

(*iv*) We have $\quad (4y - 3z)\, p + (4x - 2z)\, q = 2y - 3x.$

$\therefore$ Auxiliary equations are $\quad \frac{dx}{4y - 3z} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$ ...(1)

Taking a, b, c as multipliers, each fraction of (1)

$$= \frac{adx + bdy + cdz}{a(4y - 3z) + b(4x - 2z) + c(2y - 3x)}$$

Let $\qquad a(4y - 3z) + b(4x - 2z) + c(2y - 3x) = 0$ ...(2)

(2) $\Rightarrow \quad (4b - 3c)\, x + (4a + 2c)\, y + (-3a - 2b)\, z = 0$

Let $\qquad 4b - 3c = 0,\ 4a + 2c = 0,\ -3a - 2b = 0$

$\Rightarrow \qquad b : c = 3 : 4, \quad a : c = 1 : -2, \quad a : b = -2 : 3$

$\Rightarrow \quad a : b : c = -2 : 3 : 4$

$\therefore$ (2) is true for $a = -2,\ b = 3,\ c = 4$

$\therefore$ Each fraction of (1)

$$= \frac{-2dx + 3dy + 4dz}{-2(4y - 3z) + 3(4x - 2z) + 4(2y - 3x)} = \frac{-2dx + 3dy + 4dz}{0}$$

$\therefore$ $-2dx + 3dy + 4dz = 0$

Integrating, we have $-2x + 3y + 4z = C_1$ ...(3)

Also, (2) $\Rightarrow$ $4(ay + bx) - 3(az + cx) + 2(cy - bz) = 0$

Let $ay + bx = 0,\ az + cx = 0,\ cy - bz = 0$

$\therefore$ $a : b = -x : y,\quad a : c = -x : z,\quad b : c = y : z$

$\therefore$ $a : b : c = -x : y : z$

$\therefore$ (2) is true for $a = -x$, $b = y$, $c = z$.

$\therefore$ Each fraction of (1)

$$= \frac{-x dx + y dy + z dz}{-x(4y - 3z) + y(4x - 2z) + z(2y - 3x)} = \frac{-x dx + y dy + z dz}{0}$$

$\therefore$ $-2x dx + 2y dy + 2z dz = 0$

Integrating, we get $-x^2 + y^2 + z^2 = C_2$ ...(4)

From (3) and (4), the general solution of the given equation is

$$\mathbf{f(-2x + 3y + 4z, -x^2 + y^2 + z^2) = 0},$$

where f is any arbitrary function.

Type IV. In this type, we shall consider the solution of the equation $Pp + Qq = R$ by using the formula :

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R},$$

where P_1, Q_1, R_1 are some functions of x, y and z. If for some choice of P_1, Q_1, R_1, the sum $P_1 dx + Q_1 dy + R_1 dz$ is exact differential of a factor of $P_1 P + Q_1 Q + R_1 R$, then the quotient $\dfrac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$ is equated with a suitable fraction of auxiliary equations to get one solution of the auxiliary equations. By using different set of multipliers or by using two fractions of the auxiliary equations, we find another independent solution of the auxiliary equations.

Example 4. *Find the general solution of the following Lagrange linear equations:*

(*i*) $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

(*ii*) $(y + z)\, p + (z + x)\, q = x + y$

(*iii*) $p \cos(x + y) + q \sin(x + y) = z + \dfrac{1}{z}$

(*iv*) $(y^2 + yz + z^2)\, p + (z^2 + zx + x^2)\, q = x^2 + xy + y^2$.

Sol. (*i*) We have $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$ ...(1)

Taking 1, 1, 1 and x, y, z as multipliers, each fraction of (1)

$$= \frac{1.dx + 1.dy + 1.dz}{x^2 - yz + y^2 - zx + z^2 - xy} = \frac{x dx + y dy + z dz}{x^3 - xyz + y^3 - xyz + z^3 - xyz}$$

$$\Rightarrow \quad \frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}$$

$$\Rightarrow \quad (x+y+z)\,d(x+y+z) = \frac{1}{2}(2xdx+2ydy+2zdz)$$

$$\Rightarrow \quad (x+y+z)\,d(x+y+z) - \frac{1}{2}d(x^2+y^2+z^2) = 0$$

Integrating, we get $\frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2) = C_1$

or $$xy+yz+zx = C_1 \qquad ...(2)$$

Taking 1, – 1, 0 and 0, 1, – 1 as multipliers, each fraction of (1)

$$= \frac{dx-dy+0}{(x^2-yz)-(y^2-zx)+0} = \frac{0+dy-dz}{0+(y^2-zx)-(z^2-xy)}$$

$$\Rightarrow \quad \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

$$\Rightarrow \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

Integrating, we get $\log |x-y| - \log |y-z| = \log C_2$

or $$\left|\frac{x-y}{y-z}\right| = C_2 \quad \text{or} \quad \frac{x-y}{y-z} = \pm C_2 \qquad ...(3)$$

From (2) and (3), the general solution of the given equation is $\mathbf{f\left(xy+yz+zx, \frac{x-y}{y-z}\right) = 0}$, where f is any arbitrary function.

(*ii*) We have $(y+z)\,p + (z+x)\,q = x+y.$

$\therefore$ Auxiliary equations are $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad ...(1)$$

Taking 1, – 1, 0 and 0, 1, – 1 as multipliers, each fraction of (1)

$$= \frac{dx-dy+0}{(y+z)-(z+x)+0} = \frac{0+dy-dz}{0+(z+x)-(x+y)} \qquad ...(2)$$

$$\Rightarrow \quad \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)} \quad \Rightarrow \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0$$

Integrating, we get $\log |x-y| - \log |y-z| = \log C_1$

$$\Rightarrow \quad \left|\frac{x-y}{y-z}\right| = C_1 \quad \text{or} \quad \frac{x-y}{y-z} = \pm C_1 \qquad ...(3)$$

Taking 1, – 1, 0 and 1, 1, 1 as multipliers, each fraction of (1) and (2)

$$\frac{dx-dy+0}{(y+z)-(z+x)+0} = \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)}$$

$$\therefore \quad \frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{or} \quad 2\frac{d(x-y)}{x-y} + \frac{d(x+y+z)}{x+y+z} = 0$$

Integrating, we get $2 \log | x - y | + \log | x + y + z | = \log C_2$

$\Rightarrow \quad (x - y)^2 \,| x + y + z | = C_2 \quad \Rightarrow \quad (x - y)^2 (x + y + z) = \pm C_2$ ...(4)

From (3) and (4), the general solution of the given equation is

$$\mathbf{f}\left(\frac{\mathbf{x-y}}{\mathbf{y-z}}, (\mathbf{x-y})^2 (\mathbf{x+y+z})\right) = \mathbf{0},$$

where *f* is any arbitrary function.

(*iii*) We have $p \cos (x + y) + q \sin (x + y) = z + \dfrac{1}{z}$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{\cos (x+y)} = \dfrac{dy}{\sin (x+y)} = \dfrac{dz}{z + \frac{1}{z}}$ ...(1)

Taking 1, 1, 0 and 1, – 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx + dy + 0}{\cos (x+y) + \sin (x+y) + 0} = \frac{dx - dy + 0}{\cos (x+y) - \sin (x+y) + 0} \quad ...(2)$$

(1) and (2) $\Rightarrow \quad \dfrac{zdz}{z^2 + 1} = \dfrac{d(x+y)}{\cos (x+y) + \sin (x+y)}$

$\Rightarrow \quad \dfrac{1}{2} \cdot \dfrac{2zdz}{z^2+1} = \dfrac{dt}{\cos t + \sin t} = \dfrac{dt}{\sqrt{2} \sin (t + \pi/4)}$, where $t = x + y$.

$\Rightarrow \quad \dfrac{1}{\sqrt{2}} \cdot \dfrac{2z}{z^2+1} dz - \operatorname{cosec}\left(t + \dfrac{\pi}{4}\right) dt = 0$

Integrating, we get $\dfrac{1}{\sqrt{2}} \log |z^2 + 1| - \log \tan \dfrac{1}{2}\left(t + \dfrac{\pi}{4}\right) = \log C_1$

$$\Rightarrow \quad \frac{(z^2+1)^{1/\sqrt{2}}}{\tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right)} = C_1 \quad ...(3)$$

Also, (2) $\Rightarrow \quad \dfrac{\cos (x+y) - \sin (x+y)}{\cos (x+y) + \sin (x+y)} d(x+y) = d(x-y)$

$\Rightarrow \quad \dfrac{\cos t - \sin t}{\cos t + \sin t} dt - d(x-y) = 0$, where $t = x + y$.

Integrating, we get $\log | \cos t + \sin t | - (x - y) = \log C_2$

$\Rightarrow \quad | \cos t + \sin t | \, e^{y-x} = C_2$

$\Rightarrow \quad (\cos (x+y) + \sin (x+y))\, e^{y-x} = \pm C_2$ ...(4)

From (3) and (4), the general solution of the given equation is

$$\mathbf{f}\left(\frac{(\mathbf{z}^2+\mathbf{1})^{1/\sqrt{2}}}{\tan\left(\frac{\mathbf{x+y}}{\mathbf{2}} + \frac{\pi}{\mathbf{8}}\right)}, (\cos (\mathbf{x+y}) + \sin (\mathbf{x+y}))\, \mathbf{e}^{\mathbf{y-x}}\right) = \mathbf{0},$$

where *f* is any arbitrary function.

(*iv*) We have $(y^2 + yz + z^2)\, p + (z^2 + zx + x^2)\, q = x^2 + xy + y^2$.

$\therefore$ Auxiliary equations are $\dfrac{dx}{y^2 + yz + z^2} = \dfrac{dy}{z^2 + zx + x^2} = \dfrac{dz}{x^2 + xy + y^2}$ ...(1)

Taking 1, – 1, 0 and 0, 1, – 1 as multipliers, each fraction of (1)

$$= \frac{dx - dy + 0}{(y^2 + yz + z^2) - (z^2 + zx + x^2) + 0} = \frac{0 + dy - dz}{0 + (z^2 + zx + x^2) - (x^2 + xy + y^2)}$$

$$\Rightarrow \quad \frac{dx - dy}{y^2 - x^2 + yz - zx} = \frac{dy - dz}{z^2 - y^2 + zx - xy}$$

$$\Rightarrow \quad \frac{dx - dy}{(y - x)(y + x + z)} = \frac{dy - dz}{(z - y)(z + y + x)}$$

$$\Rightarrow \quad \frac{d(x - y)}{-(x - y)} = \frac{d(y - z)}{-(y - z)} \quad \Rightarrow \quad \frac{d(y - z)}{y - z} - \frac{d(x - y)}{x - y} = 0$$

Integrating, we get $\log | y - z | - \log | x - y | = \log C_1$

$$\Rightarrow \quad \left| \frac{y - z}{x - y} \right| = C_1 \quad \text{or} \quad \frac{y - z}{x - y} = \pm C_1 \qquad \text{...(2)}$$

Taking 1, – 1, 0 and 1, 0, – 1 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(y^2 + yz + z^2) - (z^2 + zx + x^2)} = \frac{dx - dz}{(y^2 + yz + z^2) - (x^2 + xy + y^2)}$$

$$\Rightarrow \quad \frac{dx - dy}{y^2 - x^2 + yz - zx} = \frac{dx - dz}{z^2 - x^2 + yz - xy}$$

$$\Rightarrow \quad \frac{dx - dy}{(y - x)(y + x + z)} = \frac{dx - dz}{(z - x)(z + x + y)}$$

$$\Rightarrow \quad \frac{dx - dy}{-(x - y)} = \frac{dx - dz}{-(x - z)} \quad \Rightarrow \quad \frac{d(x - z)}{x - z} - \frac{d(x - y)}{x - y} = 0$$

Integrating, we get $\log | x - z | - \log | x - y | = \log C_2$.

$$\Rightarrow \quad \left| \frac{x - z}{x - y} \right| = C_2 \quad \text{or} \quad \frac{x - z}{x - y} = \pm C_2 \qquad \text{...(3)}$$

From (2) and (3), the general solution of the given equation is $\mathbf{f\left(\dfrac{y - z}{x - y}, \dfrac{x - z}{x - y}\right) = 0}$, where f is any arbitrary function.

EXERCISE

Find the general solution of the following Lagrange linear equations:

1. (*i*) $p + q = \sin x$ (*ii*) $ap + bq = c$
(*iii*) $p \tan x + q \tan y = \tan z$ (*iv*) $yzp + zxq = xy$
(*v*) $xp + yq = z$ (*vi*) $x^2p + y^2q = z^2$
(*vii*) $(x - a)\, p + (y - b)q = z - c$ (*viii*) $y^2p - xyq = x(z - 2y)$

2. (*i*) $p - 2q = 3x^2 \sin (y + 2x)$ (*ii*) $p + 3q = z + \cot (y - 3x)$
(*iii*) $zp - zq = x + y$ (*iv*) $y^3q - xy^2p = axz$
(*v*) $xyp + y^2q + 2x^2 - xyz = 0$ (*vi*) $(p - q)\,(x + y) = z$
(*vii*) $z(xy + z^2)(px - qy) = x^4$ (*viii*) $xzp + yzq = xy$

3. (*i*) $(z-y)p+(x-z)q=y-x$ (*ii*) $x(y-z)p+y(z-x)q=z(x-y)$

(*iii*) $\left(\frac{1}{z}-\frac{1}{y}\right)p+\left(\frac{1}{x}-\frac{1}{z}\right)q=\frac{1}{y}-\frac{1}{x}$ (*iv*) $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

(*v*) $\left(\frac{b-c}{a}\right)yzp+\left(\frac{c-a}{b}\right)zxq=\left(\frac{a-b}{c}\right)xy$ (*vi*) $z(x+y)p+z(x-y)q=x^2+y^2$

(*vii*) $x(y^2-z^2)p-y(z^2+x^2)q=z(x^2+y^2)$ (*viii*) $(x-y)p+(x+y)q=2xz$

4. (*i*) $(x^2-y^2-z^2)p+2xyq=2xz$ (*ii*) $(1+y)p+(1+x)q=z$

(*iii*) $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$ (*iv*) $xzp+yzq=xy$

(*v*) $x(x+y)p-y(x+y)q+(x-y)(2x+2y+z)=0$

(*vi*) $\big(y(x+y)+az\big)p+\big(x(x+y)-az\big)q=z(x+y)$

(*vii*) $xp+zq+y=0$ (*viii*) $(x^2+y^2)p+2xyq=(x+y)z$.

Answers

1. (*i*) $f(x-y,\ z+\cos x)=0$ (*ii*) $f(bx-ay,\ cy-bz)=0$

(*iii*) $f\left(\frac{\sin x}{\sin y},\frac{\sin x}{\sin z}\right)=0$ (*iv*) $f(x^2-y^2,\ x^2-z^2)=0$

(*v*) $f\left(\frac{x}{y},\frac{y}{z}\right)=0$ (*vi*) $f\left(\frac{1}{y}-\frac{1}{x},\frac{1}{z}-\frac{1}{y}\right)=0$

(*vii*) $f\left(\frac{x-a}{y-b},\frac{y-b}{z-c}\right)=0$ (*viii*) $f(x^2+y^2,\ yz-y^2)=0$

2. (*i*) $f(2x+y,\ x^3\sin(2x+y)-z)=0$ (*ii*) $f\left(y-3x,\ x-\log|z+\cot(y-3x)|\right)=0$

(*iii*) $f(x+y,\ 2(x+y)x-z^2)=0$ (*iv*) $f\left(xy,\ \log|z|+\frac{ax}{3y^2}\right)=0$

(*v*) $f\left(\frac{x}{y},\ x-\log\left|z-\frac{2x}{y}\right|\right)=0$ (*vi*) $f(x+y,\ x-(x+y)\log|z|)=0$

(*vii*) $f(xy,\ x^4-2xyz^2-z^4)=0$ (*viii*) $f\left(xy-z^2,\frac{x}{y}\right)=0$

3. (*i*) $f(x+y+z,\ x^2+y^2+z^2)=0$ (*ii*) $f(x+y+z,\ xyz)=0$

(*iii*) $f(x+y+z,\ xyz)=0$ (*iv*) $f\left(xyz,\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$

(*v*) $f(ax^2+by^2+cz^2,\ a^2x^2+b^2y^2+c^2z^2)=0$ (*vi*) $f(2xy-z^2,\ x^2-y^2-z^2)=0$

(*vii*) $f(x^2+y^2+z^2,\ x/yz)=0$ (*viii*)

$$f\left(x+y-\log|z|,(x^2+y^2)\,e^{-2\tan^{-1}(y/x)}\right)=0$$

4. (*i*) $f\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)=0$ (*ii*) $f\left((1+x)^2-(1+y)^2,\frac{x+y+2}{z}\right)=0$

(*iii*) $f\left(z-x+y,\frac{x^2-y^2}{z^2}\right)=0$ (*iv*) $f\left(\frac{x}{y},\ xy-z^2\right)=0$

(*v*) $f\left(xy,(x+y)(x+y+z)\right)=0$ (*vi*) $f\left(\frac{x+y}{z},x^2-y^2-2az\right)=0$

(*vii*) $f\left(y^2+z^2,\frac{x}{e^{\tan^{-1}(y/z)}}\right)=0$ (*viii*) $f\left(\frac{y}{x^2-y^2},\frac{x+y}{z}\right)=0$.

NOTES

Hints

3. (*vi*) Try $x, -y, -z$ and $y, x, -z$ as multipliers.

(*vii*) Try x, y, z and $1/x, -1/y, -1/z$ as multipliers.

4. (*i*) Try x, y, z as multipliers.

(*ii*) Try 1, 1, 0 as multipliers.

(*iii*) Try 1, – 1, 0 and $x, -y$, 0 as multipliers.

(*iv*) Try $1/x$, $1/y$, 0 as multipliers.

(*v*) Try 1, 1, 0 and 1, 1, 1 as multipliers.

(*vi*) Try 1, 1, 0 and $x, -y$, 0 as multipliers.

(*vii*) We have $\dfrac{dx}{x} = \dfrac{0 + zdy - ydz}{z^2 + y^2} = \dfrac{\frac{1}{z}\,dy + \left(-\frac{y}{z^2}\right)dz}{1 + \left(\frac{y}{z}\right)^2} = \dfrac{d(y/z)}{1 + (y/z)^2}$

$$\therefore \quad \log|x| - \tan^{-1}\frac{y}{z} = \log C.$$

(*viii*) Try 1, 1, 0 and 1, – 1, 0 as multipliers.

3. PARTIAL DIFFERENTIAL EQUATINS OF THE FIRST ORDER (Equations Non-linear in *p* and *q*)

STRUCTURE

INTRODUCTION

By now we have learnt the method of solving first order partial differential equations which are linear in partial derivatives p and q. A partial differential equation of first order need not be linear in p and q. In the present chapter, we shall study the methods of solving such equations. In the first part, we shall study the method of solving some special types of equations which can be solved easily by methods other than the general method. In the second part, we shall take up **Charpit's general method** of solution.

SPECIAL TYPE I : EQUATIONS CONTAINING ONLY p AND q

Let $$g(p, q) = 0 \qquad \text{...(1)}$$

be a partial differential equation of first order and containing only p and q.

Let $$z = ax + \phi(a)\, y + c \qquad \text{...(2)}$$

be the complete solution of (1), where $\phi(a)$ is some function of a.

(2) $\Rightarrow$ $$p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \phi(a)$$

$\therefore$ (1) $\Rightarrow$ $g(a, \phi(a)) = 0$.

$\therefore$ Complete solution of (1) is

$$\mathbf{z = ax + \phi(a)\, y + c,}$$

where $g(a, \phi(a)) = 0$ and a, c are arbitrary constants.

To find the singular solution, let

$$f(x, y, z, a, c) = z - ax - \phi(a)\, y - c$$

$\therefore$ Using $f(x, y, z, a, c) = 0$, $\dfrac{\partial f}{\partial a} = 0$, $\dfrac{\partial f}{\partial c} = 0$, the singular solution is given by eliminating a and c from the equations :

$$z - ax - \phi(a)\, y - c = 0, \quad - x - \phi'(a)\, y = 0, \quad - 1 = 0$$

This is impossible, because $-1 \neq 0$.

$\therefore$ There is no singular solution.

To find the general solution, let $c = \psi(a)$, where ψ is any arbitrary function.

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by

$$\mathbf{z - ax - \phi(a)\, y - \psi(a) = 0, - x - \phi'(a)\, y - \psi'(a) = 0.}$$

SOLVED EXAMPLES

Example 1. *Solve the following partial differential equations :*

(*i*) $p^2 + q^2 = \lambda^2$ (*ii*) $p^2 - q^2 = k^2$

(*iii*) $p = 2q^2 + 1$ (*iv*) $p^2 + 6p + 2q + 4 = 0$.

Sol. (*i*) We have $p^2 + q^2 = \lambda^2$. ...(1)

This equation is of the form $g(p, q) = 0$.

Let $z = ax + \phi(a)\, y + c$ ...(2)

be the complete solution of (1), where $\phi(a)$ is some function of a.

$$(2) \Rightarrow \quad p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \phi(a)$$

$$\therefore \quad (1) \Rightarrow a^2 + (\phi(a))^2 = \lambda^2 \text{ or } \phi(a) = \pm\sqrt{\lambda^2 - a^2}$$

Let $\phi(a) = \sqrt{\lambda^2 - a^2}$ and $-\lambda \le a \le \lambda$.

$\therefore$ The complete solution is $\mathbf{z = ax + \sqrt{\lambda^2 - a^2}\; y + c}$, where a and c are arbitrary constants and $-\lambda \le a \le \lambda$. There is no singular solution.

To find the general solution, let $f(x, y, z, a, c) = z - ax - \sqrt{\lambda^2 - a^2}\; y - c$ and $c = \psi(a)$.

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations

$$\mathbf{z - ax - \sqrt{\lambda^2 - a^2}\; y - \psi(a) = 0, \quad - x + \frac{a}{\sqrt{\lambda^2 - a^2}}\, y - \psi'(a) = 0,}$$

where ψ is any arbitrary function.

(*ii*) We have $p^2 - q^2 = k^2$. ...(1)

This equation is of the form $g(p, q) = 0$.

Let $$z = ax + \phi(a)\, y + c \qquad ...(2)$$
be the complete solution of (1), where $\phi(a)$ is some function of a.

$$(2) \Rightarrow \quad p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \phi(a)$$

$$\therefore \quad (1) \Rightarrow \quad a^2 - (\phi(a))^2 = k^2 \ \text{ or } \ \phi(a) = \pm\sqrt{a^2 - k^2}$$

Let $$\phi(a) = \sqrt{a^2 - k^2} \ \text{ and } a^2 \geq k^2.$$

$\therefore$ The complete solution is $\mathbf{z = ax + \sqrt{a^2 - k^2}\, y + c}$, where a and c are arbitrary constants and $a^2 \geq k^2$.

There is no singular solution.

To find the general solution, let

$$f(x, y, z, a, c) = z - ax - \sqrt{a^2 - k^2}\, y - c \ \text{ and } c = \psi(a).$$

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0$, $\frac{\partial f}{\partial a} = 0$, the general solution is given by the equations

$$\mathbf{z - ax - \sqrt{a^2 - k^2}\, y - \psi(a) = 0, \quad -x - \frac{a}{\sqrt{a^2 - k^2}}\, y - \psi'(a) = 0,}$$

where ψ is any arbitrary function.

(*iii*) We have $$p = 2q^2 + 1. \qquad ...(1)$$

This equation is of the form $g(p, q) = 0$.

Let $$z = ax + \phi(a)\, y + c \qquad ...(2)$$
be the complete solution of (1), where $\phi(a)$ is some function of a.

$$(2) \Rightarrow \quad p = \frac{\partial z}{\partial x} = a \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \phi(a)$$

$$\therefore \quad (1) \Rightarrow \quad a = 2(\phi(a))^2 + 1 \ \text{ or } \ \phi(a) = \pm\sqrt{\frac{a-1}{2}}.$$

Let $$\phi(a) = \sqrt{\frac{a-1}{2}} \ \text{ and } a \geq 1.$$

$\therefore$ The complete solution is $\mathbf{z = ax + \sqrt{\frac{a-1}{2}}\, y + c}$, where a and c are arbitrary constants and $a \geq 1$.

There is no singular solution. To find the general solution, let

$$f(x, y, z, a, c) = z - ax - \sqrt{\frac{a-1}{2}}\, y - c \quad \text{and} \quad c = \psi(a).$$

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0$, $\frac{\partial f}{\partial a} = 0$, the general solution is given by the equations:

$$\mathbf{z - ax - \sqrt{\frac{a-1}{2}}\, y - \psi(a) = 0, \quad -x - \frac{1}{2\sqrt{2}\sqrt{a-1}}\, y - \psi'(a) = 0},$$

where ψ is any arbitrary function.

(*iv*) We have $$p^2 + 6p + 2q + 4 = 0. \qquad ...(1)$$

This equation is of the form $g(p, q) = 0$.

Let $$z = ax + \phi(a)\, y + c \quad \ldots(2)$$
be the complete solution of (1) where $\phi(a)$ is some function of a

$$(2) \Rightarrow \quad p = \frac{\partial z}{\partial x} = a \text{ and } q = \frac{\partial z}{\partial y} = \phi(a)$$

$$\therefore \ (1) \Rightarrow \quad a^2 + 6a + 2\phi(a) + 4 = 0 \text{ or } \phi(a) = -\left(\frac{a^2}{2} + 3a + 2\right)$$

$\therefore$ The complete solution is $z = ax - \left(\frac{a^2}{2} + 3a + 2\right) y + c$, where a and c are arbitrary constants.

There is no singular solution. To find the general solution, let

$$f(x, y, z, a, c) = z - ax + \left(\frac{a^2}{2} + 3a + 2\right) y - c \text{ and } c = \psi(a).$$

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0, \frac{\partial f}{\partial a} = 0,$ the general solution is given by the equations :

$$\mathbf{z - ax + \left(\frac{a^2}{2} + 3a + 2\right) y - \psi(a) = 0, - x + (a + 3)y - \psi'(a) = 0,}$$

where ψ is any arbitrary function.

WORKING STEPS FOR SOLVING g(p, q) = 0

Step I. Take complete solution as $z = ax + \phi(a)\, y + c$ where a and c are arbitrary constants.

Step II. Find $p = \frac{\partial z}{\partial x} = a, \ q = \frac{\partial z}{\partial y} = \phi(a)$.

Step III. Substitute the values of p and q in $g(p, q) = 0$ and find the value of $\phi(a)$ in terms of a. Put the value of $\phi(a)$ in the complete solution $z = ax + \phi(a)\, y + c$.

Step IV. For general solution take $f(x, y, z, a, c) = z - ax - \phi(a)y - c$, and $c = \psi(a)$. Differentiate 'f' partially w.r.t. a and write the general solution as $f(x, y, z, a, \psi(a)) = 0, \frac{\partial f}{\partial a} = 0$, *i.e.* $z - ax - \phi(a)\, y - \psi(a) = 0, - x - \phi'(a)\, y - \psi'(a) = 0$, where ψ is any arbitrary function.

Step V. Equation of the form $g(p, q) = 0$ has no singular solution.

EXERCISE A

Solving the following partial differential equations :

1. $p^2 + q^2 = 16$

2. $p = q^2$

3. $p^2 - q^2 = 1$

4. $p + q = pq$

5. $p^2 + p = q^2$

6. $p + q + pq = 0$

7. $p = e^q$

8. $p^2 + q^2 = npq$

9. $p^2q^3 = 1$

10. $pq = k$.

Answers

1. **C.S.** $z = ax - \sqrt{16 - a^2}\, y + c$, where a and c are arbitrary constants and $-4 \le a \le 4$.
   S.S. No singular solution.
   G.S. $z - ax - \sqrt{16 - a^2} - \psi(a) = 0, -x + \dfrac{a}{\sqrt{16 - a^2}} y - \psi'(a) = 0,$ where ψ is any arbitrary function.

2. **C.S.** $z = ax + \sqrt{a}\, y + c$, where a and c are arbitrary constants and $a \ge 0$.
   S.S. No singular solution.
   G.S. $z - ax - \sqrt{a}\, y - \psi(a) = 0, -x - \dfrac{1}{2\sqrt{a}} y - \psi'(a) = 0,$ where ψ is any arbitrary function.

3. **C.S.** $z = ax + \sqrt{a^2 - 1}\, y + c$, where a and c are arbitrary constants and $|a| \ge 1$.
   S.S. No singular solution.
   G.S. $z - ax - \sqrt{a^2 - 1}\, y - \psi(a) = 0, -x - \dfrac{a}{\sqrt{a^2 - 1}} y - \psi'(a) = 0$, where ψ is any arbitrary function.

4. **C.S.** $z = ax + \dfrac{a}{a-1} y + c$, where a and c are arbitrary constants and $a \ne 1$.
   S.S. No singular solution.
   G.S. $z - ax - \dfrac{a}{a-1} y - \psi(a) = 0, -x + \dfrac{1}{(a-1)^2} y - \psi'(a) = 0,$ where ψ is any arbitrary function.

5. **C.S.** $z = ax + \sqrt{a^2 + a}\, y + c$, where a and c are arbitrary constants and $a \in \mathbf{R} - (-1, 0)$.
   S.S. No singular solution.
   G.S. $z - ax - \sqrt{a^2 + a}\, y - \psi(a) = 0, -x - \dfrac{2a + 1}{2\sqrt{a^2 + a}} y - \psi'(a) = 0,$ where ψ is any arbitrary function.

6. **C.S.** $z = ax - \dfrac{a}{a+1} y + c$, where a and c are arbitrary constants and $a \ne -1$.
   S.S. No singular solution.
   G.S. $z - ax + \dfrac{a}{a+1} y - \psi(a) = 0, -x + \dfrac{1}{(a+1)^2} y - \psi'(a) = 0,$ where ψ is any arbitrary function.

7. **C.S.** $z = ax + y \log a + c$, where a and c are arbitrary constants and $a > 0$.
   S.S. No singular solution.
   G.S. $z - ax - y \log a - \psi(a) = 0, -x - \dfrac{y}{a} - \psi'(a) = 0,$ where ψ is any arbitrary function.

8. **C.S.** $z = ax + \dfrac{a}{2}\left(n + \sqrt{n^2 - 4}\right) y + c$, where a and c are arbitrary constants.
   S.S. No singular solution.
   G.S. $z - ax - \dfrac{a}{2}\left(n + \sqrt{n^2 - 4}\right) y - \psi(a) = 0, -x - \dfrac{1}{2}\left(n + \sqrt{n^2 - 4}\right) y - \psi'(a) = 0,$ where ψ is any arbitrary function.

NOTES

9. **C.S.** $z = ax + a^{-2/3}y + c$, where a and c are arbitrary constants and $a \neq 0$.

S.S. No singular solution.

G.S. $z - ax - a^{-2/3}y - \psi(a) = 0, -x + \frac{2}{3}a^{-5/3}y - \psi'(a) = 0,$ where ψ is any arbitrary function.

10. **C.S.** $z = ax + \frac{k}{a}y + c$, where a and c are arbitrary constants and $a \neq 0$.

S.S. No singular solution.

G.S. $z - ax - \frac{k}{a}y - \psi(a) = 0, -x + \frac{k}{a^2}y - \psi'(a) = 0,$ where ψ is any arbitrary function.

SPECIAL TYPE II : EQUATIONS OF THE FORM $z = px + qy + g(p, q)$

Consider the equation $\quad z = px + qy + g(p, q)$. ...(1)

Let $\quad z = ax + by + c$ ...(2)

be a solution of (1).

(2) $\Rightarrow$ $\quad p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

$\therefore$ (1) $\Rightarrow ax + by + c = ax + by + g(a, b)$

$\Rightarrow$ $\quad c = g(a, b)$

$\therefore$ (2) $\Rightarrow$ $\quad z = ax + by + g(a, b)$.

$\therefore$ Complete solution of (1) is

$z = ax + by + g(a, b)$, where a and b are arbitrary constants.

To find the singular solution, let $f(x, y, z, a, b) = z - ax - by - g(a, b)$.

$\because$ Using $f(x, y, z, a, b) = 0$, $\frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0$, the singular solution is given by eliminating a and b from the equations :

$$\mathbf{z - ax - by - g(a, b) = 0, \quad -x - \frac{\partial g}{\partial a} = 0, \quad -y - \frac{\partial g}{\partial b} = 0},$$ provided it satisfies the given equation.

To find the general solution, let $b = \psi(a)$, where ψ is any arbitrary function.

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0, \frac{\partial f}{\partial a} = 0,$ the general solution is given by

$$\mathbf{z - ax - \psi(a)y - g(a, \psi(a)) = 0, - x - \psi'(a)\,y - g'(a, \psi(a)) = 0.}$$

Remark. The partial differential equation $z = px + qy + g(p, q)$ is analogous to the **Clairaut's equation** $z = px + f(p)$. The equation $z = px + qy + g(p, q)$ is known **as extended Clairaut's equation.**

SOLVED EXAMPLES

Example 2. *Solve the following partial differential equations :*

(i) $z = px + qy + pq$ $\qquad$ *(ii)* $z = px + qy + p^2q^2$

(iii) $z = px + qy + 4\sqrt{1 + p^2 + q^2}$ $\qquad$ *(iv)* $z = px + qy + \log(pq)$.

Sol. (*i*) We have $z = px + qy + pq.$ ...(1)

This equation is of the form $z = px + qy + g(p, q).$

$\therefore$ Complete solution of (1) is $\mathbf{z = ax + by + ab}$, where a and b are arbitrary constants.

To find the singular solution, let

$$f(x, y, z, a, b) = z - ax - by - ab$$

$$\therefore \quad \frac{\partial f}{\partial a} = -x - b \quad \text{and} \quad \frac{\partial f}{\partial b} = -y - a$$

$$\therefore \quad f(x, y, z, a, b) = 0 \quad \Rightarrow \quad z - ax - by - ab = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial a} = 0 \quad \Rightarrow \quad -x - b = 0 \qquad ...(3)$$

$$\frac{\partial f}{\partial b} = 0 \quad \Rightarrow \quad -y - a = 0 \qquad ...(4)$$

Putting the values of a and b from (3) and (4) in (2), we get

$z - (-y)\,x - (-x)\,y - (-y)(-x) = 0$ or $z + xy = 0$ and it also satisfies (1).

$\therefore$ Singular solution is $\mathbf{z + xy = 0.}$

To find the general solution, let $b = \psi(a)$.

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0, \dfrac{\partial f}{\partial a} = 0,$ the general solution is given by the equations :

$\mathbf{z - ax - \psi(a)\, y - a\, \psi(a) = 0, - x - \psi'(a)\, y - \psi(a) - a\psi'(a) = 0}$, where ψ is any arbitrary function.

(*ii*) We have $z = px + qy + p^2q^2.$ ...(1)

This equation is of the form $z = px + qy + g(p, q).$

$\therefore$ Complete solution of (1) is $\mathbf{z = ax + by + a^2b^2}$, where a and b are arbitrary constants. To find the singular solution, let

$$f(x, y, z, a, b) = z - ax - by - a^2b^2$$

$$\therefore \quad \frac{\partial f}{\partial a} = -x - 2ab^2 \quad \text{and} \quad \frac{\partial f}{\partial b} = -y - 2a^2b$$

$$\therefore \quad f(x,y, z, a, b) = 0 \quad \Rightarrow \quad z - ax - by - a^2b^2 = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial a} = 0 \quad \Rightarrow \quad -x - 2ab^2 = 0 \qquad ...(3)$$

$$\frac{\partial f}{\partial b} = 0 \quad \Rightarrow \quad -y - 2a^2b = 0 \qquad ...(4)$$

$$\therefore \quad ab^2 = -\frac{x}{2}, a^2b = -\frac{y}{2}, \quad (ab)^3 = \frac{xy}{4}, \quad ab = \left(\frac{xy}{4}\right)^{1/3}$$

$$\therefore \quad (ab)b = -\frac{x}{2} \;\Rightarrow\; b = -\frac{x}{2}\cdot\left(\frac{4}{xy}\right)^{1/3} = -\frac{x^{2/3}}{2^{1/3}\,y^{1/3}} = -\left(\frac{x^2}{2y}\right)^{1/3}.$$

Similarly, $$a = -\left(\frac{y^2}{2x}\right)^{1/3}$$

$\therefore$

$$z + \left(\frac{y^2}{2x}\right)^{1/3} x + \left(\frac{x^2}{2y}\right)^{1/3} y - \left(\frac{xy}{4}\right)^{2/3} = 0 \;\Rightarrow\; z + x^{2/3}\,y^{2/3}\left(\frac{1}{2^{1/3}} + \frac{1}{2^{1/3}} - \frac{1}{2.2^{1/3}}\right) = 0$$

NOTES

$$\Rightarrow \quad \mathbf{z = -\frac{3}{2^{4/3}} x^{2/3} y^{2/3}}.$$

This gives the singular solution, because it also satisfies (1). To find the general solution, let $b = \psi(a)$.

$\therefore$ Using $f(x, y, z, a, \psi(a)) = 0, \dfrac{\partial f}{\partial a} = 0,$ the general solution is given by the equations :

$$\mathbf{z - ax - \psi(a)\, y - a^2(\psi(a))^2 = 0, - x - \psi'(a)\, y - 2a\, (\psi(a))^2 - 2a^2\psi(a)\, \psi'(a) = 0,}$$

where ψ is any arbitrary function.

(*iii*) We have $z = px + qy + 4\sqrt{1 + p^2 + q^2}$ . ...(1)

This equation is of the form $z = px + qy + g(p, q)$.

$\therefore$ Complete solution of (1) is

$\mathbf{z = ax + by + 4\sqrt{1 + a^2 + b^2}}$, where a and b are arbitrary constants.

To find the singular solution, let

$$f(x, y, z, a, b) = z - ax - by - 4\sqrt{1 + a^2 + b^2}$$

$$\therefore \quad \frac{\partial f}{\partial a} = -x - \frac{4a}{\sqrt{1 + a^2 + b^2}} \quad \text{and} \quad \frac{\partial f}{\partial b} = -y - \frac{4b}{\sqrt{1 + a^2 + b^2}}$$

$$\therefore \quad f(x, y, z, a, b) = 0 \quad \Rightarrow \quad z - ax - by - 4\sqrt{1 + a^2 + b^2} = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial a} = 0 \quad \Rightarrow \quad -x - \frac{4a}{\sqrt{1 + a^2 + b^2}} = 0 \qquad ...(3)$$

$$\frac{\partial f}{\partial b} = 0 \quad \Rightarrow \quad -y - \frac{4b}{\sqrt{1 + a^2 + b^2}} = 0 \qquad ...(4)$$

Using (3) and (4), we get $x^2 + y^2 = \dfrac{16(a^2 + b^2)}{1 + a^2 + b^2}$

$$\therefore \quad 16 - x^2 - y^2 = 16 - \frac{16(a^2 + b^2)}{1 + a^2 + b^2} = \frac{16}{1 + a^2 + b^2}$$

$$\therefore \quad \sqrt{1 + a^2 + b^2} = \frac{4}{\sqrt{16 - x^2 - y^2}}$$

$$\therefore \quad (3) \quad \Rightarrow \quad a = -x\frac{\sqrt{1 + a^2 + b^2}}{4} = -\frac{x}{4} \cdot \frac{4}{\sqrt{16 - x^2 - y^2}} = -\frac{x}{\sqrt{16 - x^2 - y^2}}.$$

Similarly, $\quad b = -\dfrac{y}{\sqrt{16 - x^2 - y^2}}$

Putting the values of a and b in (2), we get

$$z + \frac{x^2}{\sqrt{16 - x^2 - y^2}} + \frac{y^2}{\sqrt{16 - x^2 - y^2}} - \frac{16}{\sqrt{16 - x^2 - y^2}} = 0$$

or $\quad z = \dfrac{16 - x^2 - y^2}{\sqrt{16 - x^2 - y^2}} = \sqrt{16 - x^2 - y^2}$ or $\mathbf{x^2 + y^2 + z^2 = 16.}$

This gives the singular solution, because it also satisfies (1). To find the general solution, let $b = \psi(a)$.

Using $f(x, y, z, a, \psi(a)) = 0, \dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations:

$$\mathbf{z - ax - \psi(a)y - 4\sqrt{1 + a^2 + (\psi(a))^2} = 0}, \quad \mathbf{-x - \psi'(a)y - \frac{4(a + \psi(a)\,\psi'(a))}{\sqrt{1 + a^2 + (\psi(a))^2}} = 0},$$

where ψ is any arbitrary function.

(*iv*) We have $\quad z = px + qy + \log (pq)$. ...(1)

This equation is of the form $z = px + qy + g(p, q)$.

$\therefore$ Complete solution of (1) is

$\mathbf{z = ax + by + \log (ab)}$, where a and b are arbitrary constants.

To find the singular solution, let $f(x, y, z, a, b) = z - ax - by - \log (ab)$

$$\therefore \qquad \frac{\partial f}{\partial a} = -x - \frac{1}{a} \quad \text{and} \quad \frac{\partial f}{\partial b} = -y - \frac{1}{b}$$

$$\therefore \qquad f(x, y, z, a, b) = 0 \quad \Rightarrow \quad z - ax - by - \log (ab) = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial a} = 0 \quad \Rightarrow \quad -x - \frac{1}{a} = 0 \qquad ...(3)$$

$$\frac{\partial f}{\partial b} = 0 \quad \Rightarrow \quad -y - \frac{1}{b} = 0 \qquad ...(4)$$

$$\therefore \quad (3) \quad \Rightarrow \quad a = -\frac{1}{x} \quad \text{and (4)} \quad \Rightarrow \quad b = -\frac{1}{y}$$

$$\therefore \quad (2) \quad \Rightarrow \quad z + 1 + 1 - \log \frac{1}{xy} = 0 \quad \Rightarrow \quad \mathbf{z + 2 + \log xy = 0.}$$

This is the singular solution, because it also satisfies (1). To find the general solution, let $b = \psi(a)$.

Using $f(x, y, z, a, \psi(a)) = 0, \dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations:

$$\mathbf{z - ax - \psi(a)y - \log a - \log \psi(a) = 0, - x - \psi'(a)y - \frac{1}{a} - \frac{\psi'(a)}{\psi(a)} = 0},$$

where ψ is any arbitrary function.

Example 3. *Show that the complete integral of $z = px + qy - 2p - 3q$ represents all possible planes through the point (2, 3, 0). Also find the envelope of all planes represented by the complete integral, i.e., find the singular solution.*

Sol. We have $\qquad z = px + qy - 2p - 3q$. ...(1)

This equation is of the form $z = px + qy + g(p, q)$.

$\therefore$ Complete solution of (1) is $z = ax + by - 2a - 3b$, where a and b are arbitrary constants. This represents a family of planes each passing through (2, 3, 0) because $0 = a(2) + b(3) - 2a - 3b$ for all constants a and b.

To find the singular solution, let

$$f(x, y, z, a, b) = z - ax - by - 2a - 3b$$

$$\frac{\partial f}{\partial a} = 0 \Rightarrow -x-2=0, \quad \frac{\partial f}{\partial b} = 0 \Rightarrow -y-3=0$$

$\therefore$ $z - a(-2) - b(-3) - 2a - 3b = 0$ or $z = 0$. It also satisfies (1).

NOTES

$\therefore$ The singular solution is **z = 0.**

WORKING STEPS FOR SOLVING z = px + qy + g(p, q)

Step I. Take complete solution as $z = ax + by + g(a, b)$, where a and b are arbitrary constants.

Step II. For singular solution, take $f(x, y, z, a, b) = z - ax - by - g(a, b)$. Find $\frac{\partial f}{\partial a}$ and $\frac{\partial f}{\partial b}$. Eliminate a and b from the equations : $f = 0, \frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0.$ This gives the singular solution.

Step III. For general solution, take $b = \psi(a)$, where ψ is any arbitrary function. The equations : $f = 0, \frac{\partial f}{\partial a} = 0$ constitute the general solution.

EXERCISE B

Solve the following partial differential equations (Q. No. 1–10) :

1. $z = px + qy + 5pq$

2. $z = px + qy + p^2 + q^2$

3. $z = px + qy + p^2 - q^2$

4. $z = px + qy - 2p - 3q$

5. $z = px + qy + 3(pq)^{1/3}$

6. $z = px + qy + p/q$

7. $z = px + qy + 2\sqrt{pq}$

8. $z = px + qy - 2\sqrt{pq}$

9. $z = px + qy + p^2 + pq + q^2$

10. $z = px + qy + \sqrt{\alpha p^2 + \beta q^2 + 1}$.

11. Show that the complete integral of the equation $z = px + qy + \dfrac{pq}{pq - p - q}$ represents a family of planes such that the algebraic sum of the intercepts on the three coordinates axes is unity.

12. Show that the complete integral of the equation $z = px + qy + \sqrt{p^2 + q^2 + 1}$ represents a family of planes each at a unit distance from the origin.

Answers

1. **C.S.** $z = ax + by + 5ab$

S.S. $5z + xy = 0$

G.S. $z - ax - \psi(a)\, y - 5a\psi(a) = 0,\ x + 5\psi(a) + (y + 5a)\, \psi'(a) = 0$

2. **C.S.** $z = ax + by + a^2 + b^2$

S.S. $x^2 + y^2 + 4z = 0$

G.S. $z - ax - \psi(a)\, y - a^2 - (\psi(a))^2 = 0,\ x + 2a + (y + 2\psi(a))\, \psi'(a) = 0$

3. **C.S.** $z = ax + by + a^2 - b^2$

S.S. $x^2 - y^2 + 4z = 0$

G.S. $z - ax - \psi(a)\, y - a^2 + (\psi(a))^2 = 0,\ x + 2a + (y - 2\psi(a))\, \psi'(a) = 0$

4. **C.S.** $z = ax + by - 2a - 3b$

S.S. $z = 0$

G.S. $z - ax - \psi(a)\, y + 2a + 3\psi'(a) = 0,\ x + (y - 3)\, \psi'(a) - 2 = 0$

5. **C.S.** $z = ax + by + 3(ab)^{1/3}$

S.S. $xyz - 1 = 0$

G.S. $z - ax - \psi(a)\, y - 3(a\, \psi(a))^{1/3} = 0,\ x + \psi'(a)\, y + \dfrac{\psi(a) + a\psi'(a)}{(a\, \psi(a))^{2/3}} = 0$

6. **C.S.** $z = ax + by + a/b$

S.S. $xz + y = 0$

G.S. $z - ax - \psi(a)\, y - \dfrac{a}{\psi(a)} = 0,\ x + \psi'(a)\, y + \dfrac{1}{\psi(a)} - \dfrac{a\, \psi'(a)}{(\psi(a))^2} = 0$

7. **C.S.** $z = ax + by + 2\sqrt{ab}$

S.S. $(x - z)(y - z) = 1$

G.S. $z - ax - \psi(a)\, y - 2\sqrt{a\psi(a)} = 0,\ x + \psi'(a) + \dfrac{\psi(a) + a\psi'(a)}{\sqrt{a\, \psi(a)}} = 0$

8. **C.S.** $z = ax + by - 2\sqrt{ab}$

S.S. $(x - z)(y - z) = 1$

G.S. $z - ax - \psi(a)\, y + 2\sqrt{a\psi(a)} = 0,\ x + \psi'(a)\, y - \dfrac{\psi(a) + a\psi'(a)}{\sqrt{a\, \psi(a)}} = 0$

9. **C.S.** $z = ax + by + a^2 + ab + b^2$

S.S. $x^2 + y^2 - xy + 3z = 0$

G.S. $z - ax - \psi(a)\, y - a^2 - a\psi(a) - (\psi(a))^2 = 0,\ x + (y + a + 2\psi(a))\, \psi'(a) + 2a + \psi(a) = 0$

10. **C.S.** $z = ax + by + \sqrt{\alpha a^2 + \beta b^2 + 1}$

S.S. $\dfrac{x^2}{\alpha} + \dfrac{y^2}{\beta} + z^2 = 1$

G.S. $z - ax - \psi(a)\, y - \sqrt{\alpha a^2 + \beta(\psi(a))^2 + 1} = 0,\ x + \psi'(a)\, y + \dfrac{a\alpha + \beta\psi(a)\, \psi'(a)}{\sqrt{\alpha a^2 + \beta(\psi(a))^2 + 1}} = 0.$

Hint

7. **S.S.** We have $z - ax - by - 2\sqrt{ab} = 0,\ x = -\sqrt{\dfrac{b}{a}}$ and $y = -\sqrt{\dfrac{a}{b}}$.

Now $x - z = x - (ax + by + 2\sqrt{ab}) = -\sqrt{\dfrac{b}{a}} + a\sqrt{\dfrac{b}{a}} + b\sqrt{\dfrac{a}{b}} - 2\sqrt{ab} = -\sqrt{\dfrac{b}{a}}$

Similarly, $y - z = -\sqrt{\dfrac{a}{b}}$.

SPECIAL TYPE III : EQUATIONS CONTAINING ONLY z, p AND q

Let $$g(z, p, q) = 0 \qquad ...(1)$$
be a partial differential equation of first order and containing only z, p and q.

Let $z = G(u)$ where $u = x + ay$ be a solution of (1) where a is an arbitrary constant.

$$\therefore \qquad p = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}\,.\,1 = \frac{dz}{du} \quad \text{and} \quad q = \frac{dz}{du}\frac{\partial u}{\partial y} = \frac{dz}{du}\,.\,a = a\,\frac{dz}{du}$$

$$\therefore \quad (1) \quad \Rightarrow \quad g\left(z, \frac{dz}{du}, a\,\frac{dz}{du}\right) = 0$$

This is an ordinary differential equation of first order. The solution of this equation, say $f(x, y, z, a, b) = 0$ gives the complete solution of (1) where a and b are arbitrary constants. The singular solution is obtained by eliminating a and b from the equations :

$$f(x, y, z, a, b) = 0, \ \frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0, \text{ provided it satisfies the given equation.}$$

Let $b = \phi(a)$ where ϕ is an arbitrary function. The general solution is given by the equations :

$$f(x, y, z, a, \phi(a)) = 0, \ \frac{\partial f}{\partial a} = 0.$$

SOLVED EXAMPLES

Example 1. *Solve the following partial differential equations :*

(*i*) $p^2 + q^2 = 4z$ (*ii*) $z^2 (p^2 + q^2 + 1) = 1$

(*iii*) $p(1 - q^2) = q(1 - z)$.

Sol. (*i*) We have $p^2 + q^2 = 4z$. ...(1)

This equation is of the form $g(z, p, q) = 0$.

Let $z = G(u)$, where $u = x + ay$ be a solution of (1).

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\,\frac{dz}{du}.$$

$$\therefore \quad (1) \quad \Rightarrow \quad \left(\frac{dz}{du}\right)^2 + \left(a\,\frac{dz}{du}\right)^2 = 4z$$

$$\Rightarrow \qquad (1 + a^2)\left(\frac{dz}{du}\right)^2 = 4z \quad \Rightarrow \quad \frac{dz}{du} = \frac{2}{\sqrt{1+a^2}}\,z^{1/2}$$

$$\Rightarrow \qquad \frac{dz}{z^{1/2}} = \frac{2}{\sqrt{1+a^2}}\,du$$

Integrating, we get $\quad 2\sqrt{z} = \dfrac{2u}{\sqrt{1+a^2}} + b$

$$\Rightarrow \qquad 2\sqrt{1+a^2}\,\sqrt{z} = 2(x + ay) + b\sqrt{1+a^2}$$

$$\Rightarrow \qquad 4(1 + a^2)z = 4(x + ay + c)^2, \text{ where } \ 2c = b\sqrt{1+a^2}$$

$\Rightarrow$ $\mathbf{(1 + a^2)\, z - (x + ay + c)^2 = 0.}$

This is the complete solution.

Let $\qquad f(x, y, z, a, c) = (1 + a^2)\, z - (x + ay + c)^2$

$$\therefore \qquad \frac{\partial f}{\partial a} = 2az - 2(x + ay + c)y \quad \text{and} \quad \frac{\partial f}{\partial c} = -\,2(x + ay + c)$$

$\therefore \quad f(x, y, z, a, c) = 0 \Rightarrow (1 + a^2)z - (x + ay + c)^2 = 0$ ...(2)

$\frac{\partial f}{\partial a} = 0 \Rightarrow 2az - 2(x + ay + c) = 0$ ...(3)

$\frac{\partial f}{\partial c} = 0 \Rightarrow -2(x + ay + c) = 0$ ...(4)

(4) $\Rightarrow x + ay + c = 0$

$\therefore$ (3) $\Rightarrow 2az - 2(0)y = 0 \Rightarrow 2az = 0 \Rightarrow \mathbf{z = 0.}$

This is the singular solution, because $z = 0$ also satisfies (1).

Let $c = \phi(a)$. Using $f(x, y, z, a, \phi(a)) = 0$, $\frac{\partial f}{\partial a} = \mathbf{0}$, the general solution is given by

$$\mathbf{(1 + a^2)\, z - (x + ay + \phi(a))^2 = 0,\; 2az - 2(x + ay + \phi(a))\,(y + \phi'(a)) = 0},$$

where ϕ is any arbitrary function.

(*ii*) We have $z^2(p^2 + q^2 + 1) = 1.$ ...(1)

This equation is of the form $g(z, p, q) = 0$.

Let $z = G(u)$, where $u = x + ay$ be a solution of (1).

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{dx} \text{ and } q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}.$$

$$\therefore \quad (1) \Rightarrow z^2\left(\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right) = 1$$

$$\Rightarrow \quad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2} - 1 \Rightarrow \frac{dz}{du} = \frac{\pm 1}{\sqrt{1 + a^2}}\frac{\sqrt{1 - z^2}}{z}$$

Let $$\frac{dz}{du} = \frac{1}{\sqrt{1 + a^2}}\frac{\sqrt{1 - z^2}}{z}.$$

$$\therefore \quad \frac{z}{\sqrt{1 - z^2}}dz = \frac{du}{\sqrt{1 + a^2}}$$

Integrating, we get $$-\sqrt{1 - z^2} = \frac{u}{\sqrt{1 + a^2}} + b$$

$$\Rightarrow \quad -\sqrt{1 + a^2}\sqrt{1 - z^2} = u + c, \text{ where } c = b\sqrt{1 + a^2}$$

$$\Rightarrow \quad \mathbf{(1 + a^2)\,(1 - z^2) = (x + ay + c)^2.}$$

This is the complete solution.

Let $f(x, y, z, a, c) = (1 + a^2)(1 - z^2) - (x + ay + c)^2$

$$\therefore \quad \frac{\partial f}{\partial a} = 2a(1 - z^2) - 2(x + ay + c)y \quad \text{and} \quad \frac{\partial f}{\partial c} = -2(x + ay + c)$$

$\therefore \quad f(x, y, z, a, c) = 0 \Rightarrow (1 + a^2)(1 - z^2) - (x + ay + c)^2 = 0$ ...(2)

$\frac{\partial f}{\partial a} = 0 \Rightarrow 2a(1 - z^2) - 2(x + ay + c)y = 0$ ...(3)

$\frac{\partial f}{\partial c} = 0 \Rightarrow -2(x + ay + c) = 0$ ...(4)

(4) $\Rightarrow x + ay + c = 0$

$\therefore$ (3) $\Rightarrow 2a(1 - z^2) - 2(0)y = 0$

NOTES

$$\Rightarrow \quad 2a(1-z^2) = 0 \quad \Rightarrow \quad 1 - z^2 = 0 \quad \Rightarrow \quad \mathbf{z^2 - 1 = 0.}$$

This is the singular solution, because $z^2 - 1 = 0$ also satisfies (1).

Let $c = \phi(a)$. Using $f(x, y, z, a, c) = 0$, $\frac{\partial f}{\partial a} = \mathbf{0}$, the general solution is given by

$$\mathbf{(1 + a^2)(1 - z^2) - (x + ay + \phi(a))^2 = 0, 2a(1 - z^2) - 2(x + ay + \phi(a))(y + \phi'(a)) = 0,}$$

where ϕ is any arbitrary function.

(*iii*) We have $\qquad p(1 - q^2) = q(1 - z)$. ...(1)

This equation is of the form $g(z, p, q) = 0$.

Let $z = G(u)$, where $u = x + ay$ be a solution of (1).

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{dx} \text{ and } q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

$$\therefore \quad (1) \quad \Rightarrow \quad \frac{dz}{du}\left(1 - a^2\left(\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(1 - z)$$

$$\Rightarrow \qquad \frac{dz}{du} = 0 \qquad \text{...(2)}$$

or

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a(1 - z) \qquad \text{...(3)}$$

(2) $\Rightarrow$ $z = c$. This is not a complete solution because it does not contain two arbitrary constants.

$$(3) \quad \Rightarrow \qquad \left(\frac{dz}{du}\right)^2 = \frac{1 - a + az}{a^2}.$$

Let $$\frac{dz}{du} = \frac{\sqrt{1 - a + az}}{a}$$

$$\therefore \qquad (1 - a + az)^{-1/2}\, dz = \frac{du}{a}$$

Integrating, we get $$\frac{(1 - a + az)^{1/2}}{(1/2)\, a} = \frac{u}{a} + b$$

$$\Rightarrow \qquad 2\sqrt{1 - a + az} = u + ab$$

$$\Rightarrow \qquad \mathbf{4(1 - a + az) = (x + ay + c)^2}, \text{ where } c = ab.$$

This is the complete solution.

Let $$f(x, y, z, a, c) = 4(1 - a + az) - (x + ay + c)^2$$

$$\therefore \qquad \frac{\partial f}{\partial a} = 4(-1 + z) - 2(x + ay + c)y \quad \text{and} \quad \frac{\partial f}{\partial c} = -2(x + ay + c)$$

$$\therefore \qquad f(x, y, z, a, c) = 0 \quad \Rightarrow \quad 4(1 - a + az) - (x + ay + c)^2 = 0 \qquad \text{...(4)}$$

$$\frac{\partial f}{\partial a} = 0 \qquad \Rightarrow \qquad 4(-1 + z) - 2(x + ay + c)y = 0 \qquad \text{...(5)}$$

$$\frac{\partial f}{\partial c} = 0 \quad \Rightarrow \quad -2(x + ay + c) = 0 \qquad \text{...(6)}$$

(6) $\Rightarrow$ $x + ay + c = 0$

$\therefore$ (5) $\Rightarrow$ $4(-1 + z) - 2(0)y = 0 \Rightarrow 4(-1 + z) = 0$

$\Rightarrow$ $-1 + z = 0 \Rightarrow z = 1$

$\therefore$ (4) $\Rightarrow$ $4(1 - a + a(1)) - (0)^2 = 0$

$\Rightarrow$ $4 = 0$, which is impossible.

$\therefore$ There is no singular solution. Let $c = \phi(a)$. Using $f(x, y, z, a, \phi(a)) = 0$ and $\frac{\partial f}{\partial a} = 0$, the general solution is given by $\mathbf{4(1 - a + az) - (x + ay + \phi(a))^2 = 0, - 4 + 4z - 2(x + ay + \phi(a))\,(y + \phi'(a)) = 0,}$ where ϕ is any arbitrary function.

WORKING STEPS FOR SOLVING g(z, p, q) = 0

Step I. Take $z = G(u)$, where $u = x + ay$.

Step II. By putting $p = \frac{dz}{du}$ and $q = a\,\frac{dz}{du}$, the given equation reduces to an ordinary differential equation of first order. Let its solution be $f(x, y, z, a, b) = 0$. This gives the complete solution of the given equation.

Step III. For singular solution, eliminate a and b from the equations : $f = 0$, $\frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0.$

Step IV. For general solution, take $b = \phi(a)$, where ϕ is any arbitrary function. The equations : $f = 0$, $\frac{\partial f}{\partial a} = 0$ constitute the general solution.

EXERCISE C

Solve the following partial differential equations :

1. $p^2 + q^2 = z$

2. $z^2(p^2 + q^2 + 2) = 1$

3. $p^2 + pq = 4z$

4. $pz = 1 + q^2$

5. $z = pq$

6. $9(p^2z + q^2) = 4$

7. $p^3 + q^3 = 3pqz, z > 0$

8. $p^3 + q^3 = 27z$

9. $4(1 + z^3) = 9z^4\, pq$

10. $q^2 = z^2p^2(1 - p^2)$.

Answers

1. **C.S.** $4(1 + a^2)\, z = (x + ay + c)^2$

S.S. $z = 0$

G.S. $4(1 + a^2)\, z - (x + ay + \phi(a))^2 = 0,\ 8az - 2(x + ay + \phi(a))(y + \phi'(a)) = 0$

2. **C.S.** $(1 + a^2)(1 - 2z^2) = 4(x + ay + c)^2$

S.S. $1 - 2z^2 = 0$

G.S. $(1 + a^2)(1 - 2z^2) - 4(x + ay + \phi(a))^2 = 0,\ a(1 - 2z^2) - 4(x + ay + \phi(a))(y + \phi'(a)) = 0$

3. **C.S.** $(1 + a)z = (x + ay + c)^2$

S.S. $z = 0$

G.S. $(1 + a)z - (x + ay + \phi(a))^2 = 0,\ z - 2(x + ay + \phi(a))(y + \phi'(a)) = 0$

NOTES

4. **C.S.** $z^2 - z\sqrt{z^2 - 4a^2} + 4a^2 \log(z + \sqrt{z^2 - 4a^2}) = 4(x + ay + b)$

S.S. There is no singular solution.

G.S. $z^2 - z\sqrt{z^2 - 4a^2} + 4a^2 \log(z + \sqrt{z^2 - 4a^2}) - 4(x + ay + \phi(a)) = 0,$

$$\frac{az}{\sqrt{z^2 - 4a^2}} + 2a \log\left(z + \sqrt{z^2 - 4a^2}\right) - \frac{4a^3}{(z + \sqrt{z^2 - 4a^2})\sqrt{z^2 - 4a^2}} - y - \phi'(a) = 0.$$

5. **C.S.** $4az = (x + ay + b)^2$

S.S. $z = 0$

G.S. $4az - (x + ay + \phi(a))^2 = 0,\ 2z - (x + ay + \phi(a))\ (y + \phi'(a)) = 0.$

6. **C.S.** $(z + a^2)^3 = (x + ay + b)^2$

S.S. No singular solution.

G.S. $(z + a^2)^3 - (x + ay + \phi(a))^2 = 0,\ 3a(z + a^2)^2 - (x + ay + \phi(a))(y + \phi'(a)) = 0$

7. **C.S.** $(1 + a^3) \log z = 3a(x + ay) + b$

S.S. No singular solution.

G.S. $(1 + a^3) \log z - 3a(x + ay) - \phi(a) = 0,\ 3a^2 \log z - 3x - 6ay - \phi'(a) = 0.$

8. **C.S.** $(1 + a^3)\ z^2 = 8(x + ay + b)^3$

S.S. $z = 0$

G.S. $(1 + a^3)\ z^2 - 8(x + ay + \phi(a))^3 = 0,\ a^2z^2 - 8(x + ay + \phi(a))^2(y + \phi'(a)) = 0.$

9. **C.S.** $a(1 + z^3) = (x + ay + b)^2$

S.S. $z^3 + 1 = 0$

G.S. $a(1 + z^3) - (x + ay + \phi(a))^2 = 0,\ 1 + z^3 - 2(x + ay + \phi(a))(y + \phi'(a)) = 0$

10. **C.S.** $z^2 = (x + ay + b)^2 + a^2$

S.S. $z = 0$

G.S. $z^2 - (x + ay + \phi(a))^2 - a^2 = 0,\ (x + ay + \phi(a))(y + \phi'(a)) + a = 0.$

SPECIAL TYPE IV : EQUATIONS OF THE FORM $f_1(x, p) = f_2(y, q)$

Consider the equation $f_1(x, p) = f_2(y, q)$. ...(1)

Let each side of (1) be equal to a.

$\therefore$ $f_1(x, p) = a$ ...(2) $f_2(y, q) = a$ ...(3)

Solving (2) for p, let $p = F_1(x, a)$

Solving (3) for q, let $q = F_2(y, a)$.

Since z is a function of x and y, we have $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\ dx + q\ dy$

$\therefore$ $dz = F_1(x, a)\ dx + F_2(y, a)\ dy$

Integrating, we get $\mathbf{z = \int F_1(x, a)\ dx + \int F_2(y, a)\ dy + b.}$

This represents the complete solution of the given equation.

To find the singular solution, let

$$f(x, y, z, a, b) = z - \int F_1(x, a)\, dx - \int F_2(y, a)\, dy - b\ .$$

$\therefore$ Using $f(x, y, z, a, b) = 0$, $\dfrac{\partial f}{\partial a} = 0, \dfrac{\partial f}{\partial b} = 0$, the singular solution is given by eliminating a and b from the equations :

$$z - \int F_1(x,a)\,dx - \int F_2(y,a)\,dy - b = 0,$$

$$\frac{\partial}{\partial a}\left(\int F_1(x,a)\,dx + \int F_2(y,a)\,dy\right) = 0 \text{ and } -1 = 0.$$

This is impossible, because $-1 \neq 0$.

$\therefore$ There is no singular solution.

To find the general solution, let $b = \phi(a)$, where ϕ is an arbitrary function. Using $f(x, y, z, a, \phi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations :

$$\mathbf{z - \int F_1(x, a)\,dx - \int F_2(y,a)\,dy - \phi(a) = 0,}$$

$$\mathbf{\frac{\partial}{\partial a}\left(-\int F_1(x,a)\,dx - \int F_2(y,a)\,dy\right) - \phi'(a) = 0.}$$

SOLVED EXAMPLES

Example 5. *Solve the following partial differential equations :*

(*i*) $\sqrt{p} - \sqrt{q} + 3x = 0$ (*ii*) $py + qx + pq = 0$

(*iii*) $p^2y\,(1 + x^2) = qx^2$ (*iv*) $px + q = p^2$.

Sol. (*i*) We have $\sqrt{p} + 3x = \sqrt{q}$. ...(1)

This equation is of the form $f_1(x, p) = f_2(y, q)$.

Let each side of (1) be equal to a.

$\therefore$ $\sqrt{p} + 3x = a$ ...(2) $\sqrt{q} = a$ ...(3)

(2) $\Rightarrow$ $p = (a - 3x)^2$ and (3) $\Rightarrow$ $q = a^2$

Now $dz = p\,dx + q\,dy$

$\therefore$ $dz = (a - 3x)^2\,dx + a^2\,dy$

Integrating, we get $z = \int (a - 3x)^2\,dx + \int a^2\,dy + b.$

$$\Rightarrow \qquad \mathbf{z = \frac{(a - 3x)^3}{-9} + a^2y + b.}$$

This is the complete solution. There is no singular solution.

To find the general solution, let $f(x, y, z, a, b) = z + \dfrac{1}{9}(a - 3x)^3 - a^2y - b$ and $b = \phi(a)$.

$\therefore$ Using $f(x, y, z, a, \phi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$.

The general solution is given by the equations :

$$\mathbf{z + \frac{1}{9}(a - 3x)^3 - a^2y - \phi(a) = 0,\ \frac{1}{3}(a - 3x)^2 - 2ay - \phi'(a) = 0,}$$

where ϕ is any arbitrary function.

NOTES

(*ii*) We have $py + qx + pq = 0.$

$$\Rightarrow \quad py + q(x + p) = 0 \quad \Rightarrow \quad \frac{p}{x+p} = -\frac{q}{y} \quad \text{...(1)}$$

This equation is of the form $f_1(x, p) = f_2(y, q)$. Let each side of (1) be equal to a.

$$\therefore \quad \frac{p}{x+p} = a \quad \text{...(2)} \qquad -\frac{q}{y} = a \quad \text{...(3)}$$

$$(2) \Rightarrow \quad p = \frac{ax}{1-a} \quad \text{and} \quad (3) \Rightarrow \quad q = -ay$$

Now $dz = p\,dx + q\,dy$

$$\therefore \quad dz = \frac{ax}{1-a}\,dx - ay\,dy$$

Integrating, we get $z = \int \frac{ax}{1-a}\,dx - \int ay\,dy + \frac{b}{2}.$

$$\Rightarrow \quad z = \frac{a}{1-a}\cdot\frac{x^2}{2} - \frac{ay^2}{2} + \frac{b}{2}$$

$$\Rightarrow \quad \mathbf{2z = \frac{a}{1-a}x^2 - ay^2 + b.}$$

This is the complete solution.

There is no singular solution. To find the general solution, let

$$f(x, y, z, a, b) = 2z - \frac{a}{a-1}x^2 + ay^2 - b \quad \text{and} \quad b = \phi(a).$$

$\therefore$ Using $f(x, y, z, a, \phi(a)) = 0,\quad \frac{\partial f}{\partial a} = 0$,

The general solution is given by the equations :

$\mathbf{2z - \frac{a}{1-a}x^2 + ay^2 - \phi(a) = 0,}\quad \mathbf{-\frac{x^2}{(1-a)^2} + y^2 - \phi'(a) = 0}$, where ϕ is any arbitrary func-tion.

(*iii*) We have $p^2y\,(1 + x^2) = qx^2.$

$$\Rightarrow \quad \frac{1+x^2}{x^2}p^2 = \frac{q}{y} \quad \text{...(1)}$$

This equation is of the form $f_1(x, p) = f_2(y, q)$.

Let each side of (1) be equal to a.

$$\therefore \quad \frac{1+x^2}{x^2}p^2 = a \quad \text{...(2)} \qquad \frac{q}{y} = a \quad \text{...(3)}$$

$$(2) \Rightarrow \quad p = \pm\frac{\sqrt{a}\,x}{\sqrt{1+x^2}} \quad \text{and} \quad (3) \Rightarrow \quad q = ay \qquad (\text{Assuming } a \geq 0)$$

Let $p = \frac{\sqrt{a}x}{\sqrt{1+x^2}}$

Now $dz = p\,dx + q\,dy$

$\therefore \qquad dz = \dfrac{\sqrt{a}\, x}{\sqrt{1+x^2}}\, dx + ay\, dy$

Integrating, we get $\quad z = \dfrac{\sqrt{a}}{2} \displaystyle\int \dfrac{2x}{\sqrt{1+x^2}}\, dx + a \int y\, dy + \dfrac{b}{2}.$

$\Rightarrow \qquad z = \sqrt{a}\, \sqrt{1+x^2} + \dfrac{ay^2}{2} + \dfrac{b}{2}$

$\Rightarrow \qquad \mathbf{2z = 2\sqrt{a}\, \sqrt{1+x^2} + ay^2 + b.}$

This is the complete solution. There is no singular solution.

To find the general solution, let $f(x, y, z, a, b) = 2z - 2\sqrt{a}\sqrt{1+x^2} - ay^2 - b$ and $b = \phi(a)$.

$\therefore$ Using $f(x, y, z, a, \phi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations :

$$2z - 2\sqrt{a}\sqrt{1+x^2} - ay^2 - \phi(a) = 0, \quad -2 \,.\, \frac{1}{2\sqrt{a}} \,.\, \sqrt{1+x^2} - y^2 - \phi'(a) = 0$$

or $\qquad \mathbf{2z - 2\sqrt{a}\sqrt{1+x^2} - ay^2 - \phi(a) = 0, \qquad \sqrt{1+x^2} + \sqrt{a}y^2 + \sqrt{a}\, \phi'(a) = 0,}$

where ϕ is any arbitrary function.

(*iv*) We have $\qquad px + q = p^2.$

$\Rightarrow \qquad p^2 - px = q \qquad \ldots(1)$

This equation is of the form $\quad f_1(x, p) = f_2(y, q)$.

Let each side of (1) be equal to a.

$\therefore \qquad p^2 - px = a \qquad \ldots(2) \qquad q = a \qquad \ldots(3)$

(2) $\qquad \Rightarrow \quad p^2 - px - a = 0 \quad \Rightarrow \quad p = \dfrac{1}{2}\left(x \pm \sqrt{x^2 + 4a}\right)$

Let $\qquad p = \dfrac{1}{2}\left(x + \sqrt{x^2 + 4a}\right)$

Now $\qquad dz = p\, dx + q\, dy$

$\therefore \qquad dz = \dfrac{1}{2}\left(x + \sqrt{x^2 + 4a}\right) dx + a\, dy$

Integrating, we get $\quad z = \dfrac{x^2}{4} + \dfrac{1}{2}\left[\dfrac{x\sqrt{x^2 + 4a}}{2} + \dfrac{4a}{2} \log\left| x + \sqrt{x^2 + 4a} \right|\right] + ay + b$

or $\qquad \mathbf{z = \dfrac{1}{4}\left(x^2 + x\sqrt{x^2 + 4a}\right) + a \log\left| x + \sqrt{x^2 + 4a} \right| + ay + b.}$

This is the complete solution. There is no singular solution.

To find the general solution, let

$f(x, y, z, a, b) = z - \dfrac{1}{4}(x^2 + x\sqrt{x^2 + 4a}) - a \log (x + \sqrt{x^2 + 4a}) - ay - b$ and $b = \phi(a)$.

NOTES

$\therefore$ Using $f(x, y, z, a, \phi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$, the general solution is given by the equations :

$$z - \frac{1}{4}(x^2 + x\sqrt{x^2 + 4a}) - a \log (x + \sqrt{x^2 + 4a}) - ay - \phi(a) = 0,$$

$$-\frac{x}{4} \cdot \frac{4}{2\sqrt{x^2 + 4a}} - \log (x + \sqrt{x^2 + 4a}) - a \cdot \frac{1}{x + \sqrt{x^2 + 4a}} \left(0 + \frac{4}{2\sqrt{x^2 + 4a}}\right) - y - \phi'(a) = 0$$

or
$$\mathbf{z - \frac{1}{4}(x^2 + x\sqrt{x^2 + 4a}) - a \log (x + \sqrt{x^2 + 4a}) - ay - \phi(a) = 0,}$$

$$\mathbf{\frac{x}{2\sqrt{x^2 + 4a}} + \log (x + \sqrt{x^2 + 4a}) + \frac{2a}{(x + \sqrt{x^2 + 4a})\sqrt{x^2 + 4a}} + y + \phi'(a) = 0,}$$

where ϕ is any arbitrary function.

WORKING STEPS FOR SOLVING $f_1(x, p) = f_2(y, q)$

Step I. Take each side of $f_1(x, p) = f_2(y, q)$ equal to a.

Step II. Solve equations for p and q. Let $p = F_1(x, a)$, $q = F_2(y, a)$. Write $z = p\, dx + q\, dy$ and substitute the values of p and q. Integrate this equation to get the complete solution.

Step III. Equation of the form $f_1(x, p) = f_2(y, q)$ has no singular solution.

Step IV. Take the complete solution as $f(x, y, z, a, b) = 0$. Put $b = \phi(a)$. The general solution is given by the equations : $f(x, y, z, a, \phi(a)) = 0$, $\dfrac{\partial f}{\partial a} = 0$.

EXERCISE D

Solve the following partial differential equations :

1. $p - q = x^2 + y^2$

2. $x(1 + y)\, p = y(1 + x)\, q$

3. $pq = xy$

4. $q = xyp^2$

5. $x^2p^2 = q^2y$

6. $q(p - \cos x) = \cos y$

7. $yp = 2yx + \log q$

8. $\sqrt{p} + \sqrt{q} = 2x$

9. $p^2 - x = q^2 - y$

10. $p - 3x^2 = q^2 - y$.

Answers

1. **C.S.** $z = \frac{1}{3}(x^3 - y^3) + a(x + y) + b$

S.S. No singular solution

G.S. $z - \frac{1}{3}(x^3 - y^3) - a(x + y) - \phi(a) = 0, x + y + \phi'(a) = 0$

2. **C.S.** $z = a \log xy + a(x + y) + b$

S.S. No singular solution

G.S. $z - a \log xy - a(x + y) - \phi(a) = 0$, $\log xy + x + y + \phi'(a) = 0$

3. **C.S.** $2z = ax^2 + y^2/a + b$

S.S. No singular solution

G.S. $2z - ax^2 - y^2/a - \phi(a) = 0$, $x^2 - y^2/a^2 + \phi'(a) = 0$

NOTES

4. **C.S.** $2z = 4\sqrt{ax} + ay^2 + b$

S.S. No singular solution

G.S. $2z - 4\sqrt{ax} - ay^2 - \phi(a) = 0, 2\sqrt{x/a} + y^2 + \phi'(a) = 0$

5. **C.S.** $z = \sqrt{a} \log x + 2\sqrt{ay} + b$

S.S. No singular solution

G.S. $z - \sqrt{a} \log x - 2\sqrt{ay} - \phi(a) = 0, \log x + 2\sqrt{y} + 2\sqrt{a}\, \phi'(a) = 0$

6. **C.S.** $z = ax + \sin x + \frac{1}{a} \sin y + b$

S.S. No singular solution

G.S. $z - ax - \sin x - \frac{1}{a} \sin y - \phi(a) = 0, x - \frac{1}{a^2} \sin y + \phi'(a) = 0$

7. **C.S.** $z = x^2 + ax + \frac{e^{ay}}{a} + b$

S.S. No singular solution

G.S. $z - x^2 - ax - \frac{e^{ay}}{a} - \phi(a) = 0, x + \frac{e^{ay}(ay - 1)}{a^2} + \phi'(a) = 0$

8. **C.S.** $z = \frac{1}{6}(2x - a)^3 + a^2 y + b$

S.S. No singular solution

G.S. $z - \frac{1}{6}(2x - a)^3 - a^2 y - \phi(a) = 0$, $4ay - (2x - a)^2 + 2\phi'(a) = 0$

9. **C.S.** $3z = 2(x + a)^{3/2} + 2(y + a)^{3/2} + b$

S.S. No singular solution

G.S. $3z - 2(x + a)^{3/2} - 2(y + a)^{3/2} - \phi(a) = 0,\ 3\sqrt{x + a} + 3\sqrt{y + a} + \phi'(a) = 0$

10. **C.S.** $z = ax + x^3 + \frac{2}{3}(a + y)^{3/2} + b$

S.S. No singular solution

G.S. $z - ax - x^3 - \frac{2}{3}(a + y)^{3/2} - \phi(a) = 0, x + \sqrt{a + y} + \phi'(a) = 0.$

USE OF TRANSFORMATIONS

At times the use of transformations helps a lot in changing a partial differential equation to a much simpler form.

Remark. Keeping in view the scope of the present book, we are restricting ourselves only to the finding of complete solutions of partial differential equations which are reducible to the form $g(P, Q) = 0$, where $P = \frac{\partial Z}{\partial X}, Q = \frac{\partial Z}{\partial Y}$.

SOLVED EXAMPLES

Example 6. *Find the complete solution of the following differential equations with the help of transformations :*

(i) $x^2p^2 + y^2q^2 = z$ (ii) $x^2p^2 + y^2q^2 = 4z^2$

(iii) $pq = x^m y^n z^{2l}$ (iv) $(1 - x^2)\, yp^2 + x^2 q = 0.$

NOTES

Sol. (*i*) We have $$x^2p^2 + y^2q^2 = z. \quad ...(1)$$

(1) $\Rightarrow$ $$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$\Rightarrow$ $$\left(\frac{z^{-1/2}}{x^{-1}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-1/2}}{y^{-1}}\frac{\partial z}{\partial y}\right)^2 = 1 \quad ...(2)$$

Let X, Y, Z be new variables such that $dX = x^{-1}\,dx$, $dY = y^{-1}\,dy$, $dZ = z^{-1/2}\,dz$.

$\therefore$ By using integration, we have X = log x, Y = log y, Z = $2\sqrt{z}$

$\therefore$ $$P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = \frac{1}{\sqrt{z}}\cdot\frac{\partial z}{\partial x}\cdot x = \frac{z^{-1/2}}{x^{-1}}\frac{\partial z}{\partial x}$$

and $$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial y}\cdot\frac{dy}{dY} = \frac{1}{\sqrt{z}}\cdot\frac{\partial z}{\partial y}\cdot y = \frac{z^{-1/2}}{y^{-1}}\frac{\partial z}{\partial y}.$$

$\therefore$ (2) $\Rightarrow$ $P^2 + Q^2 = 1$ ...(3)

This equation is of the form $g(P, Q) = 0$.

Let $$Z = aX + \phi(a)Y + c \quad ...(4)$$

be the complete solution of (3), where $\phi(a)$ is some function of a.

(4) $\Rightarrow$ $$P = \frac{\partial Z}{\partial X} = a \text{ and } Q = \frac{\partial Z}{\partial Y} = \phi(a)$$

$\therefore$ (3) $\Rightarrow$ $a^2 + (\phi(a))^2 = 1$ or $\phi(a) = \pm\sqrt{1-a^2}$

Let $\phi(a) = \sqrt{1-a^2}$, $-1 \le a \le 1$.

$\therefore$ The complete solution of (3) is $Z = aX + \sqrt{1-a^2}\;Y + c$.

$\therefore$ The complete solution of (1) is $\mathbf{2\sqrt{z} = a \log x + \sqrt{1-a^2} \log y + c}$, where a and c are arbitrary constants.

(*ii*) We have $$x^2p^2 + y^2q^2 = 4z^2. \quad ...(1)$$

(1) $\Rightarrow$ $$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 4$$

$\Rightarrow$ $$\left(\frac{z^{-1}}{x^{-1}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-1}}{y^{-1}}\frac{\partial z}{\partial y}\right)^2 = 4 \quad ...(2)$$

Let X, Y, Z be new variables such that $dX = x^{-1}\,dx$, $dY = y^{-1}\,dy$, $dZ = z^{-1}\,dz$

$\therefore$ By using integration, we have X = log x, Y = log y, Z = log z

$\therefore$ $$P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = \frac{1}{z}\cdot\frac{\partial z}{\partial x}\cdot x = \frac{z^{-1}}{x^{-1}}\frac{\partial z}{\partial x}$$

and $$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial y}\cdot\frac{dy}{dY} = \frac{1}{z}\cdot\frac{\partial z}{\partial y}\cdot y = \frac{z^{-1}}{y^{-1}}\frac{\partial z}{\partial y}.$$

$\therefore$ (2) $\Rightarrow$ $P^2 + Q^2 = 4$ ...(3)

This equation is of the form $g(P, Q) = 0$.

Let $$Z = aX + \phi(a)Y + c \quad ...(4)$$

be the complete solution of (3), where $\phi(a)$ is some function of a.

(4) $\Rightarrow$ $$P = \frac{\partial Z}{\partial X} = a \text{ and } Q = \frac{\partial Z}{\partial Y} = \phi(a)$$

$\therefore$ (3) $\Rightarrow$ $a^2 + (\phi(a))^2 = 4$ or $\phi(a) = \pm\sqrt{4-a^2}$

NOTES

Let $\phi(a) = \sqrt{4 - a^2}$, $-2 \le a \le 2$.

$\therefore$ The complete solution of (3) is $Z = aX + \sqrt{4-a^2}\ Y + c$.

$\therefore$ The complete solution of (1) is $\mathbf{\log z = a \log x + \sqrt{4-a^2}\ \log y + c}$, where a and c are arbitrary constants.

(*iii*) We have $pq = x^m y^n z^{2l}$. ...(1)

$$\Rightarrow \quad \frac{z^{-2l}}{x^m y^n}\, pq = 1 \quad \Rightarrow \quad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1 \qquad ...(2)$$

Let X, Y, Z be new variables such that $dX = x^m\, dx$, $dY = y^n\, dy$, $dZ = z^{-l}\, dz$

$\therefore$ By using integration, we have $X = \dfrac{x^{m+1}}{m+1}$, $Y = \dfrac{y^{n+1}}{n+1}$, $Z = \dfrac{z^{1-l}}{1-l}$.

$$\therefore \quad P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial x}\cdot\frac{dx}{dX} = z^{-l}\cdot\frac{\partial z}{\partial x}\cdot\frac{1}{x^m} = \frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}$$

and
$$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{dz}\cdot\frac{\partial z}{\partial y}\cdot\frac{dx}{dY} = z^{-l}\frac{\partial z}{\partial y}\frac{1}{y^n} = \frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}.$$

$\therefore$ (2) $\Rightarrow$ $PQ = 1$ ...(3)

This equation is of the form $g(P, Q) = 0$.

Let $Z = aX + \phi(a)\, Y + c$ ...(4)

be the complete solution of (3), where $\phi(a)$ is some function of a.

$$(4) \Rightarrow \quad P = \frac{\partial Z}{\partial X} = a \quad \text{and} \quad Q = \frac{\partial Z}{\partial Y} = \phi(a)$$

$\therefore$ (3) $\Rightarrow$ $a\,\phi(a) = 1$ or $\phi(a) = 1/a$

$\therefore$ The complete solution of (3) is $Z = aX + \dfrac{1}{a} Y + c$.

$\therefore$ The complete solution of (1) is $\mathbf{\dfrac{z^{1-l}}{1-l} = \dfrac{a}{m+1} x^{m+1} + \dfrac{1}{(n+1)\,a} y^{n+1} + c}$, where a and c are arbitrary constants and $a \neq 0$.

(*iv*) We have $(1 - x^2)\, yp^2 + x^2 q = 0$. ...(1)

$$(1) \Rightarrow \quad \frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{y}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \left(\frac{1}{x/\sqrt{1-x^2}}\frac{\partial z}{\partial x}\right)^2 + \frac{1}{y}\frac{\partial z}{\partial y} = 0 \qquad ...(2)$$

Let X, Y, Z be new variables such that $dX = \dfrac{x}{\sqrt{1-x^2}}dx$, $dY = y\, dy$, $dZ = dz$

NOTES

$\therefore$ By using integration, we have

$$X = -\frac{1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} = -\sqrt{1-x^2}\,, \quad Y = \frac{y^2}{2}\,, \quad Z = z.$$

$$\therefore \qquad P = \frac{\partial Z}{\partial X} = \frac{dZ}{dz} \cdot \frac{\partial z}{\partial x} \cdot \frac{dx}{dX} = 1 \cdot \frac{\partial z}{\partial x} \cdot \frac{\sqrt{1-x^2}}{x} = \frac{1}{x/\sqrt{1-x^2}} \frac{\partial z}{\partial x}$$

and

$$Q = \frac{\partial Z}{\partial Y} = \frac{dZ}{dz} \cdot \frac{\partial z}{\partial y} \cdot \frac{dy}{dY} = 1 \cdot \frac{\partial z}{\partial y} \cdot \frac{1}{y} = \frac{1}{y} \frac{\partial z}{\partial y}.$$

$\therefore$ (2) $\Rightarrow$ $P^2 + Q = 0$ ...(3)

This equation is of the form $g(P, Q) = 0$.

Let $Z = aX + \phi(a)\, Y + c$ ...(4)

be the complete solution of (3), where $\phi(a)$ is some function of a.

(4) $\Rightarrow$ $P = \dfrac{\partial Z}{\partial X} = a$ and $Q = \dfrac{\partial Z}{\partial Y} = \phi(a)$

$\therefore$ (3) $\Rightarrow$ $a^2 + \phi(a) = 0$ or $\phi(a) = -a^2$

$\therefore$ The complete solution of (3) is $Z = aX - a^2Y + c$.

$\therefore$ The complete solution (1) is $\mathbf{z = -a\sqrt{1-x^2} - \dfrac{a^2}{2} y^2 + c}$, where a and c are arbitrary constants.

EXERCISE E

Find the complete solution of the following partial differential equations :

1. $p^2x + q^2y = z$
2. $pq = x^m y^n z^l$
3. $p^2 - q^2 = z$
4. $p^2 + q^2 = z$
5. $p^2y\,(1 + x^2) = qx^2$
6. $zy^2p = x(y^2 + z^2q^2)$
7. $z^2(p^2/x^2 + q^2/y^2) = 1$
8. $x^4p^2 + y^2zq - z^2 = 0$
9. $p^m \sec^{2m} x + z^l q^n \operatorname{cosec}^{2n} y = z^{lm/(m-n)}$
10. $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0.$

Answers

1. $\sqrt{z} = a\sqrt{x} \pm \sqrt{1-a^2}\,\sqrt{y} + c,\ -1 \le a \le 1$
2. $\dfrac{2z^{1-(l/2)}}{2-l} = \dfrac{a}{m+1} x^{m+1} + \dfrac{1}{(n+1)\,a} y^{n+1} + c,\ a \ne 0$
3. $2\sqrt{z} = ax \pm \sqrt{a^2-1}\, y + c,\ |a| \ge 1$
4. $2\sqrt{z} = ax \pm \sqrt{1-a^2}\, y + c,\ -1 \le a \le 1$
5. $z = a\sqrt{1+x^2} + \dfrac{1}{2}a^2y^2 + c$
6. $z^2 = ax^2 \pm \sqrt{a-1}\, y^2 + c,\ a \ge 1$
7. $z^2 = ax^2 \pm \sqrt{1-a^2}\, y^2 + c,\ -1 \le a \le 1$
8. $xy \log z + ay = (a^2 - 1)\, x + cxy$
9. $\dfrac{m-n}{m-n-l} z^{(m-n-l)/(m-n)} = \dfrac{a}{4}(2x + \sin 2x) + \dfrac{(1-a^m)^{1/n}}{4}(2y - \sin 2y) + c$
10. $z^2 = ax^2 - \left(\dfrac{a^2}{2} + 3a + 2\right) y^2 + c.$

NOTES

CHARPIT'S GENERAL METHOD OF SOLUTION

If the given partial differential equation is not of any of the given special types, then the given equation is solved by using Charpit's general method.

Let $f(x, y, z, p, q) = 0$ ...(1)

be the given partial differential equation of first order and non-linear in p and q.

Since z is a function of x and y, we have $dz = p\,dx + q\,dy$ ...(2)

The procedure is to first find an equation involving x, y, z, p, q.

Let $F(x, y, z, p, q) = 0$ ...(3)

be the required equation involving x, y, z, p, q. The equations (1) and (3) are solved to find the values of p and q. The values of p and q are substituted in (2) and is then integrated to get the desired result.

Differentiating (1) and (3) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad ...(4)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad ...(5)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad ...(6)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad ...(7)$$

Multiplying (4) by $\frac{\partial F}{\partial p}$, (6) by $\frac{\partial f}{\partial p}$ and subtracting, we get

$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p} + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad ...(8)$$

Multiplying (5) by $\frac{\partial F}{\partial q}$, (7) by $\frac{\partial F}{\partial q}$ and subtracting, we get

$$\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q} + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad ...(9)$$

Adding (8) and (9), we get

$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p} + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q} + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q = 0$$

$$\left(\because \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial p}{\partial q}\right)$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$

$$+ \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0 \qquad ...(10)$$

This is a Lagrange equation of first order with independent variables x, y, z, p, q and dependent variable F. The auxiliary equations of (10) are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \qquad ...(11)$$

Any of the integrals of (11) satisfy (10). If any such integral involve p or q or both, it can be taken as the assumed relation (3). Simpler the integral involving p or q or both, that is derived from (11), the easier will be the solution of (1). Substituting these values of p and q in $dz = p\,dx + q\,dy$ and then integrating, we find the required solution.

SOLVED EXAMPLES

Example 7. *Find the complete solution of the following partial differential equations by using Charpit's method :*

(*i*) $z = px + qy + p^2 + q^2$ (*ii*) $z^2 = pqxy$

(*iii*) $px + qy = pq$ (*iv*) $(p^2 + q^2)\,y = qz$

(*v*) $p = (qy + z)^2$.

Sol. (*i*) We have $z = px + qy + p^2 + q^2$. ...(1)

Let $f(x, y, z, p, q) = z - px - qy - p^2 - q^2$.

$$\therefore \quad \frac{\partial f}{\partial x} = -p, \quad \frac{\partial f}{\partial y} = -q, \quad \frac{\partial f}{\partial z} = 1, \quad \frac{\partial f}{\partial p} = -x - 2p, \quad \frac{\partial f}{\partial q} = -y - 2q$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

$\Rightarrow$

$$\frac{dp}{-p + p(1)} = \frac{dq}{-q + q(1)} = \frac{dz}{-p(-x - 2p) - q(-y - 2q)} = \frac{dx}{-x - 2p} = \frac{dy}{-y - 2q}$$

$$\Rightarrow \quad \frac{dp}{0} = \frac{dq}{0} = \frac{dz}{px + qy + 2p^2 + 2q^2} = \frac{dx}{-x - 2p} = \frac{dy}{-y - 2q}$$

First fraction implies $dp = 0$. Let $p = a$.

Similarly, let $q = b$.

Now $dz = p\,dx + q\,dy \quad \therefore \quad dz = a\,dx + b\,dy$

Integrating, we get $z = ax + by + c$.

Putting the value of z in (1), we get $ax + by + c = ax + by + a^2 + b^2$ or $c = a^2 + b^2$.

$$\therefore \quad \mathbf{z = ax + by + a^2 + b^2.}$$

This is the complete solution.

(*ii*) We have $z^2 = pqxy$. ...(1)

Let $f(x, y, z, p, q) = z^2 - pqxy$.

$$\therefore \quad \frac{\partial f}{\partial x} = -pqy, \quad \frac{\partial f}{\partial y} = -pqx, \quad \frac{\partial f}{\partial z} = 2z, \quad \frac{\partial f}{\partial p} = -qxy, \quad \frac{\partial f}{\partial q} = -pxy$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \quad \frac{dp}{-pqy+p(2z)} = \frac{dq}{-pqx+q(2z)} = \frac{dz}{-p(-qxy)-q(-pxy)}$$

$$= \frac{dx}{-(-qxy)} = \frac{dy}{-(-pxy)}$$

$$\Rightarrow \quad \frac{dp}{2pz-pqy} = \frac{dq}{2qz-pqx} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy} \quad \ldots(2)$$

$$(2) \Rightarrow \quad \frac{xdp+pdx}{x(2pz-pqy)+p(qxy)} = \frac{ydq+qdy}{y(2qz-pqx)+q(pxy)}$$

$$\Rightarrow \quad \frac{x\,dp+p\,dx}{2xpz} = \frac{y\,dq+q\,dy}{2yqz} \quad \Rightarrow \quad \frac{d(xp)}{xp} = \frac{d(yq)}{yq}$$

Integrating, we get $\log xp = \log yq + a$

$$\Rightarrow \quad xp = yqb^2 \quad \ldots(3) \quad (\text{Putting } a = \log b^2)$$

Solving (1) and (3) for p and q, we get $p = \frac{bz}{x}$ and $q = \frac{z}{by}$.

Now $dz = p\,dx + q\,dy \quad \therefore \quad dz = \frac{bz}{x}dx + \frac{z}{by}dy$

$$\Rightarrow \quad \frac{dz}{z} = \frac{b}{x}dx + \frac{1}{by}dy$$

Integrating, we get $\log z = b \log x + \frac{1}{b} \log y + \log c$.

$$\therefore \quad \mathbf{z = cx^b\, y^{1/b}}$$

This is the complete solution.

(*iii*) We have $px + qy = pq$. ...(1)

Let $f(x, y, z, p, q) = px + qy - pq$.

$$\therefore \quad \frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \quad \frac{dp}{p+p(0)} = \frac{dq}{q+q(0)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

$$\Rightarrow \quad \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px-qy+2pq} = \frac{dx}{q-x} = \frac{dy}{p-y} \quad \ldots(2)$$

$$(2) \Rightarrow \quad \frac{dp}{p} = \frac{dq}{q} \quad \Rightarrow \quad \log p = \log q + \log a \quad \Rightarrow \quad p = aq \quad \ldots(3)$$

Solving (1) and (3), we get $p = 0, q = 0$ or $p = ax + y, q = \frac{ax+y}{a}$.

Case I. p = 0, q = 0

$$\therefore \quad dz = p\,dx + q\,dy \quad \Rightarrow \quad dz = 0\,dx + 0\,dy = 0 \quad \Rightarrow \quad z = c.$$

This is not a complete solution, because it does not contain two arbitrary constants.

NOTES

Case II. $\mathbf{p = ax + y, q = \dfrac{ax + y}{a}}$

$$\therefore \quad dz = p\,dx + q\,dy \;\Rightarrow\; dz = (ax + y)\,dx + \left(\frac{ax + y}{a}\right)dy$$

$$= \frac{ax + y}{a}(a\,dx + dy) = \frac{ax + y}{a}\,d\,(ax + y)$$

Integrating, we get $z = \mathbf{\dfrac{(ax + y)^2}{2a} + b.}$

This is the complete solution.

(*iv*) We have $(p^2 + q^2)\,y = qz.$ ...(1)

Let $f(x, y, z, p, q) = p^2y + q^2y - qz.$

$$\therefore \quad \frac{\partial f}{\partial x} = 0,\quad \frac{\partial f}{\partial y} = p^2 + q^2,\quad \frac{\partial f}{\partial z} = -q,\quad \frac{\partial f}{\partial p} = 2py,\quad \frac{\partial f}{\partial q} = 2qy - z$$

Charpit's auxiliary equations are

$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}}$$

$$\Rightarrow \quad \frac{dp}{0 + p(-q)} = \frac{dq}{p^2 + q^2 + q(-q)} = \frac{dz}{-p(2py) - q(2qy - z)} = \frac{dx}{-2py} = \frac{dy}{-(2qy - z)}$$

$$\Rightarrow \quad \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{z - 2qy} \qquad ...(2)$$

$$(2) \;\Rightarrow\; \frac{dp}{-pq} = \frac{dq}{p^2} \;\Rightarrow\; \frac{dp}{-q} = \frac{dq}{p} \;\Rightarrow\; p\,dp + q\,dq = 0$$

Integrating, we get $\dfrac{p^2}{2} + \dfrac{q^2}{2} = \dfrac{a}{2}$ or $p^2 + q^2 = a$ ...(3)

Solving (1) and (3), we get $p = \sqrt{a - \dfrac{a^2y^2}{z^2}}$ and $q = \dfrac{ay}{z}$.

$$\therefore \quad dz = p\,dx + q\,dy \;\Rightarrow\; dz = \frac{\sqrt{az^2 - a^2y^2}}{z}dx + \frac{ay}{z}dy$$

$$\Rightarrow \quad z\,dz - ay\,dy = \sqrt{az^2 - a^2y^2}\;dx$$

$$\Rightarrow \quad d\left(\frac{z^2}{2}\right) - d\left(\frac{ay^2}{2}\right) = \sqrt{az^2 - a^2y^2}\,dx$$

$$\Rightarrow \quad \frac{1}{2a}d\,(az^2 - a^2y^2) = \sqrt{az^2 - a^2y^2}\,dx \;\Rightarrow\; \frac{d(az^2 - a^2y^2)}{\sqrt{az^2 - a^2y^2}} = 2a\,dx$$

Integrating, we get $2\sqrt{az^2 - a^2y^2} = 2ax + 2b \;\Rightarrow\; \mathbf{az^2 - a^2y^2 = (ax + b)^2.}$

This is the complete solution.

(*v*) We have $p = (qy + z)^2.$ ...(1)

Let $f(x, y, z, p, q) = (qy + z)^2 - p$

$$\therefore \quad \frac{\partial f}{\partial x} = 0,\quad \frac{\partial f}{\partial y} = 2(qy + z)q,\quad \frac{\partial f}{\partial z} = 2(qy + z),\quad \frac{\partial f}{\partial p} = -1,\quad \frac{\partial f}{\partial q} = 2(qy + z)\,y$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \quad \frac{dp}{0+p2(qy+z)}=\frac{dq}{2q(qy+z)+q2(qy+z)}=\frac{dz}{-p(-1)-q2y(qy+z)}=\frac{dx}{-(-1)}$$

$$=\frac{dy}{-2y(qy+z)}$$

$$\Rightarrow \quad \frac{dp}{2p(qy+z)}=\frac{dq}{4q(qy+z)}=\frac{dz}{p-2yq(qy+z)}=\frac{dx}{1}=\frac{dy}{-2y(qy+z)} \quad \ldots(2)$$

$$(2) \Rightarrow \quad \frac{dp}{2p(qy+z)}=\frac{dy}{-2y(qy+z)} \quad \Rightarrow \quad \frac{dp}{p}+\frac{dy}{y}=0$$

Integrating, we get $\log|p| + \log|y| = \log C$

$$\Rightarrow \quad |py| = C \;\Rightarrow\; py = \pm C \;\Rightarrow\; py = a \qquad (\text{Putting } a = \pm C)$$

$$\therefore \quad p = a/y$$

Putting the value of p in (1), we get $\frac{a}{y} = (qy + z)^2$.

$$\Rightarrow \quad qy+z=\sqrt{\frac{a}{y}} \;\Rightarrow\; qy=\sqrt{\frac{a}{y}}-z \;\Rightarrow\; q=\frac{\sqrt{a}}{y^{3/2}}-\frac{z}{y}$$

$$\therefore \quad dz = p\,dx + q\,dy \;\Rightarrow\; dz=\frac{a}{y}dx+\left(\frac{\sqrt{a}}{y^{3/2}}-\frac{z}{y}\right)dy$$

$$\Rightarrow \quad ydz = adx + \left(\frac{\sqrt{a}}{\sqrt{y}}-z\right)dy \;\Rightarrow\; ydz + zdy = adx + \frac{\sqrt{a}}{\sqrt{y}}dy$$

$$\Rightarrow \quad d(yz) = adx + \frac{\sqrt{a}}{\sqrt{y}}dy$$

Integrating, we get $yz = ax + \sqrt{a}\,.\,\frac{y^{1/2}}{1/2} + b \;\Rightarrow\; \mathbf{yz = ax + 2\sqrt{ay} + b.}$

This is the complete solution.

WORKING STEPS FOR USING CHARPIT'S METHOD

Step I. Shift all terms of the given equation to the left side and denote the left side by $f(x, y, z, p, q)$.

Step II. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$.

Step III. Write the Charpit's auxiliary equations and substitute the values of partial derivatives of f and simplify.

Step IV. Select any two fractions so that the resulting integral is the simplest relation involving at least one of p and q. This relation and the given equation are solved to find the values of p and q.

Step V. Put the values of p and q in the equation $dz = p\,dx + q\,dy$ and integrate. This gives the complete solution of the given equation.

NOTES

EXERCISE F

Find the complete solution of the following partial differential equations by using Charpit's method :

1. $q = px + q^2$

2. $q = 3p^2$

3. $p^2 - y^2q = y^2 - x^2$

4. $pxy + pq + qy = yz$

5. $2(z + px + qy) = yp^2$

6. $2z + p^2 + qy + 2y^2 = 0$

7. $q = px + p^2$

8. $2xz - px^2 - 2qxy + pq = 0$

9. $p(1 + q^2) + (b - z)q = 0$

10. $(p^2 + q^2)x = pz$.

Answers

1. $z = (a - a^2) \log x + ay + b$

2. $z = ax + 3a^2y + b$

3. $z = \dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a} - \dfrac{a^2}{y} - y + b$

4. $(z - ax)(y + a)^a = be^y$

5. $yz - \dfrac{ax}{y} + \dfrac{a^2}{4y^2} = b$

6. $y^2((a - x)^2 + 2z + y^2) = b$

7. $z = -\dfrac{x^2}{4} \pm \left[\dfrac{x\sqrt{x^2 + 4a}}{4} + a \log (x + \sqrt{x^2 + 4a})\right] + ay + b$

8. $z = ay + b(x^2 - a)$

9. $2\sqrt{c(z - b) - 1} = x + cy + a$

10. $az^2 - a^2x^2 = (ay + b)^2$.

Hints

4. $p = a \quad \Rightarrow \quad q = \dfrac{y(z - ax)}{a + y}$.

$\therefore \quad dz = p\, dx + q\, dy \quad \Rightarrow \quad dz = a\, dx + \dfrac{y(z - ax)}{a + y} dy \quad \Rightarrow \quad \dfrac{dz - a\, dx}{z - ax} = \dfrac{y}{a + y} dy$.

5. Charpit's auxiliary equations implies $\dfrac{dp}{4p} = \dfrac{dy}{-2y}$ or $py^2 = a$

Also $$dz = \frac{a}{y^2} dx + \left(-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right) dy$$

$$\Rightarrow \quad (y\, dz + z\, dy) - a\left(\frac{1}{y} dx - \frac{x\, dy}{y^2}\right) - \frac{a^2}{2y^3} dy = 0.$$

6. $dz = (a - x)\, dx - \dfrac{1}{y}(2z + 2y^2 + (a - x)^2)\, dy$

Multiplying by $2y^2$, we get $(2y^2\, dz + 4yz\, dy) = (2y^2(a - x)\, dx - 2y(a - x)^2\, dy) - 4y^3\, dy$.

UNIT II

4. HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

STRUCTURE

INTRODUCTION

Till now we have been discussing the methods of solving partial differential equations of the first order. A partial differential equation of the first order involves, only the first order partial derivatives (p and q) of the dependent variable z. Now we shall consider the solution of partial differential equations of order higher than one.

PARTIAL DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

We know that the order of a partial differential equation is the order of the highest partial derivative occurring in the given partial differential equation.

NOTES

For example, $\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = x^2 + y$ is a partial differential equation of order 3. For the sake of simplicity, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are denoted by D(or D_x) and D′(or D_y) respectively. Thus, the above differential equation can also be written as $(D^3 + 2DD' + D')z = x^2 + y$ or as $(D_x^3 + 2D_x D_y + D_y)z = x^2 + y$.

Remark. DD′z stands for $\frac{\partial^2 z}{\partial x \partial y}$ and not for $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$. The product $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$ is denoted as (Dz)(D′z).

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

We know that a partial differential equation is called linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied together. A linear partial differential equation of order n is of the form

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \ldots\ldots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$

$$+ \ldots\ldots + \left(M \frac{\partial z}{\partial x} + N \frac{\partial z}{\partial y}\right) + Pz = F(x, y),$$

where the coefficients $A_0, A_1, \ldots\ldots$ N, P are constants or functions of x and y. If the coefficients $A_0, A_1, \ldots\ldots$, N, P are all constants then such a differential equation is called a **linear partial differential equation with constant coefficients**.

In a linear partial differential equation, the orders of various partial derivatives occurring in the equation may or may not be equal. In case the orders of all partial derivatives involved in the equation are same then it is called a **homogeneous linear partial differential equation** and otherwise it is called a **non-homogeneous linear partial differential equation.**

A homogeneous linear partial differential equation with constant coefficients is of the form $A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + A_n \frac{\partial^n z}{\partial y^n} = F(x, y)$,

where $A_0, A_1, \ldots\ldots, A_n$ are all constants.

Consider the following partial differential equations :

(*i*) $4\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^x$

(*ii*) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$

(*iii*) $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = \sin x$

(*iv*) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = e^{-x}$

(*v*) $(2DD' + D'^2 - 3D)z = 3\cos(3x - 2y)$

(*vi*) $r - s + 2q - z = x^2y^2$

(*vii*) $xyr + x^2s - yt = x^3e^y$

(*viii*) $r + (y/x)s = 15xy^2$

(*ix*) $yt + xs + q = 8yx^2 + 9y^2$.

Differential equations (*i*), (*ii*) and (*iii*) are homogeneous linear partial differential equations with constant coefficients.

Differential equations (*iv*), (*v*) and (*vi*) non-homogeneous linear partial differential equation with constant coefficients.

Differential equations (*vii*) and (*viii*) are homogeneous linear partial differential equations with variable coefficients.

Differential equation (*ix*) is a non-homogeneous linear partial differential equation with variable coefficients.

In the present chapter, we shall consider the methods of solving homogeneous linear partial differential equations with constant coefficients.

SOME THEOREMS

Let $f(D, D')z = F(x, y)$ be a linear partial differential equation with constant coefficients.

$\therefore$ The function $f(D, D')$ is of the form

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + ...+ A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + ...+ B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$
$$+ ...... + \left(M \frac{\partial z}{\partial x} + N \frac{\partial z}{\partial y}\right) + Pz$$

where $A_0, A_1, ..., N, P$ are all constants.

Theorem 1. Let f(D, D′)z = 0 be a linear partial differential equation with constant coefficients. If $u_1, u_2, ..., u_m$ be m solutions of f(D, D′)z = 0, then prove that $\sum_{i=1}^{m} c_i u_i$ is also a solution of f(D, D′)z = 0.

Proof. u_i is a solution of $f(D, D')\, z = 0$ for $1 \le i \le m$.

$\therefore \qquad f(D, D')\, u_i = 0 \quad \text{for } 1 \le i \le m$

Now $f(D, D')\left(\sum_{i=1}^{m} c_i u_i\right) = \sum_{i=1}^{m} f(D, D')(c_i u_i) = \sum_{i=1}^{m} c_i f(D, D') u_i = \sum_{i=1}^{m} c_i\,(0) = 0.$

$\therefore \qquad \sum_{i=1}^{m} c_i u_i$ is also a solution of the equation $F(D, D')z = 0$.

Theorem 2. Let f(D, D′)z = F(x, y) be a linear partial differential equation with constant coefficients. If u is a solution of f(D, D′)z = 0 and v is a solution of f(D, D′) z = F(x, y), then prove that u + v is a solution of f(D, D′) z = F(x, y).

Proof. u is a solution of $\quad f(D, D')z = 0.$

$\therefore \qquad f(D, D')u = 0 \qquad ...(1)$

v is a solution of $\quad f(D, D')z = F(x, y).$

$\therefore \qquad f(D, D')v = F(x, y) \qquad ...(2)$

Now
$$f(D, D')\,(u + v) = f(D, D')u + f(D, D')v$$
$$= 0 + F(x, y) \qquad \text{(Using (1), (2))}$$
$$= F(x, y).$$

$\therefore \qquad u + v$ is a solution of $f(D, D')z = F(x, y)$.

NOTES

GENERAL SOLUTION OF HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION f(D, D′)z = 0 WITH CONSTANT COEFFICIENTS

Let $$f(D, D')z = A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + ...+ A_n \frac{\partial^n z}{\partial y^n} = 0 \qquad ...(1)$$

be a homogeneous linear partial differential equation with constant coefficients, where $A_0 \neq 0$.

(1) can also be written as $$(A_0D^n + A_1D^{n-1}D' + ... + A_nD'^n)\, z = 0 \qquad ...(2)$$

Let (2) be equivalent to $A_0[(D - m_1D')(D - m_2D') ...... (D - m_nD')]\, z = 0$ ...(3)

Treating D and D′ as variables, the equations (2) and (3) implies

$$A_0\left(\frac{D}{D'}\right)^n + A_1\left(\frac{D}{D'}\right)^{n-1} + ... + A_n = A_0\left(\frac{D}{D'} - m_1\right)\left(\frac{D}{D'} - m_2\right)......\left(\frac{D}{D'} - m_n\right)$$

∴ $m_1, m_2, ..., m_n$ are roots of the equation

$$A_0m^n + A_1m^{n-1} + ... + A_n = 0. \qquad ...(4)$$

The equation (4) is called the **auxiliary equation** of (2). This equation can be obtained by putting D equal to m and D′ equal to '1' in the operator of equation (2) and equating it to zero.

The roots $m_1, m_2, ..., m_n$ of the auxiliary equation may or may not be distinct.

Case I. Roots are distinct.

∴ $m_1 \neq m_2 \neq m_3 \neq ...... \neq m_n$.

Equation (3) shows that for $1 \leq i \leq n$, the solution of $(D - m_iD')\, z = 0$ is a solution of (3) and hence of (1).

$(D - m_iD')\, z = 0 \quad \Rightarrow \quad p - m_i\, q = 0$

$\Rightarrow \quad \frac{dx}{1} = \frac{dy}{-m_i} = \frac{dz}{0} \quad \Rightarrow \quad dy + m_i\, dx = 0, \quad dz = 0$

$\Rightarrow \quad y + m_ix = c_1, \quad z = c_2$

$\Rightarrow \quad z = \phi_i(y + m_ix).$ (Putting $c_2 = \phi_i(c_1)$)

∴ $z = \phi_1(y + m_1x), z = \phi_2(y + m_2x), ... , z = \phi_n(y + m_n x)$ are solutions of (1).

∴ $z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + ... + \phi_n(y + m_nx)$ is also a solution of (1). Since this solution contains n arbitrary functions $\phi_1, \phi_2, ..., \phi_n$, this solution represents the general solution of the given equation.

Case II. Roots are not distinct.

Let $m_1 = m_2 \neq m_3 \neq ...... \neq m_n$. In this case, the solution of (1) can be written as

$z = (\phi_1 + \phi_2)(y + m_1x) + ... + \phi_n(y + m_nx).$ ($\because \ m_1 = m_2$)

This solution contains only $n - 1$ arbitrary functions $\phi_1 + \phi_2, \phi_3, ..., \phi_n$.

∴ This is not a general solution.

Using $m_1 = m_2$ in (3), we get, $A_0\, [(D - m_1D')^2\, (D - m_3D')\, ...\, (D - m_nD')]\, z = 0$ ...(5)

Equation (5) shows that for $3 \leq i \leq n$, the solution of $(D - m_iD')z = 0$ is also a solution of (5) and hence of (1).

∴ $z = \phi_i(y + m_ix), \quad 3 \leq i \leq n$ is a solution of (1).

$\therefore \qquad z = \phi_3(y + m_3x) + ... + \phi_n(y + m_nx)$ ...(6)

is a solution of (1).

The solution of $(D - m_1D')^2z = 0$ is also a solution of (5) and hence of (1).

$$(D - m_1D')^2 z = 0 \Rightarrow (D - m_1D')(D - m_1D') z = 0 \quad ...(7)$$

Let $\quad (D - m_1D') z = u.$ ...(8)

$\therefore$ (7) $\Rightarrow \quad (D - m_1D')u = 0$

$\Rightarrow \quad u = \psi_1(y + m_1x)$, where ψ_1 is arbitrary.

$\therefore$ (7) $\Rightarrow \quad p - m_1q = \psi_1(y + m_1x)$

$$\Rightarrow \quad \frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\psi_1(y + m_1x)} \quad ...(9)$$

(9) $\Rightarrow \quad dy + m_1dx = 0 \Rightarrow y + m_1x = c$

Taking the first and third fractions of (9), we get

$$dx = \frac{dz}{\psi_1(c)} \quad \text{or} \quad dz = \psi_1(c)\, dx$$

$$\Rightarrow \quad z = \psi_1(c)\, x + d$$

$$\Rightarrow \quad z = x\psi_1(y + m_1x) + \psi_2(y + m_2x)$$

...(10) (Putting $d = \psi_2(c)$)

Combining (6) and (10),

$$z = x\psi_1(y + m_1x) + \psi_2(y + m_2x) + \phi_3(y + m_3x) + ... + \phi_n(y + m_nx)$$

is also a solution of (1).

Since this solution contains n arbitrary functions $\psi_1, \psi_2, \phi_3, ..., \phi_n$, this solution represents the general solution of the given equation.

Remarks 1. If the root m_1 of the auxiliary equation is repeated r times, then the corresponding part of the general solution is $\phi_1(y + m_1x) + x\phi_2(y + m_1x) + ... + x^{r-1}\phi_r(y + m_1x)$, where $\phi_1, \phi_2, ..., \phi_r$ are arbitrary functions.

2. The auxiliary equation of a homogeneous linear partial differential equation with constant coefficients is obtained by putting $D = m$ and $D' = 1$ in the operator of the equation and then equating it to zero.

Exceptional Case.

If $A_0 = 0$, $A_1 \neq 0$, then equation (2) becomes $(A_1D^{n-1}D' + ... + A_nD'^n)\, z = 0$

or $\quad D'(A_1D^{n-1} + A_2D^{n-2}D' + ... + A_nD'^{\,n-1})z = 0$ ...(1)

$\therefore$ The solution of $D'z = 0$ is also a solution of (1).

$$D'z = 0 \Rightarrow \frac{\partial z}{\partial y} = 0 \Rightarrow z = \phi(x), \text{ where } \phi \text{ is arbitrary.}$$

Similarly, if D'^r is a factor of the operator of the equation then the corresponding part of the general solution is $z = \phi_1(x) + y\phi_2(x) + ... + y^{r-1}\phi_r(x)$, where $\phi_1, \phi_2, ..., \phi_r$ are arbitrary functions.

SOLVED EXAMPLES

Example 1. *Find the general solution of the following partial differential equations :*

(*i*) $\dfrac{\partial^2 z}{\partial x^2} - 8\dfrac{\partial^2 z}{\partial x \partial y} + 15\dfrac{\partial^2 z}{\partial y^2} = 0$ (*ii*) $2r + 5s + 2t = 0$

(*iii*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\, z = 0$ (*iv*) $(D^4 + D'^4 - 2D^2D'^2)\, z = 0$

(*v*) $(D^3D'^2 + D^2D'^3)\, z = 0.$

Sol. (*i*) We have $\frac{\partial^2 z}{\partial x^2} - 8\frac{\partial^2 z}{\partial x \partial y} + 15\frac{\partial^2 z}{\partial y^2} = 0$.

$\Rightarrow \quad (D^2 - 8DD' + 15D'^2)\, z = 0$ ...(1)

By putting D = m and D′ = 1 in the operator of (1), the auxiliary equation of (1) is

$$m^2 - 8m + 15 = 0.$$

$\therefore \quad m = 3, 5$

$\therefore$ The general solution of the given equation is $\mathbf{z = \phi_1(y + 3x) + \phi_2(y + 5x)}$, where ϕ_1, ϕ_2 are arbitrary functions.

(*ii*) We have $2r + 5s + 2t = 0$.

$\therefore \quad (2D^2 + 5DD' + 2D'^2)\, z = 0$ ...(1)

The auxiliary equation of (1) is $2m^2 + 5m + 2 = 0$.

$\therefore \quad m = -1/2, -2$

$\therefore$ The general solution of the given equation is

$$\mathbf{z = \phi_1\left(y - \frac{1}{2}x\right) + \phi_2(y - 2x)},$$ where ϕ_1, ϕ_2 are arbitrary functions.

(*iii*) We have $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$. ...(1)

The auxiliary equation of (1) is $m^3 - 6m^2 + 11m - 6 = 0$.

$\therefore \quad m = 1, 2, 3$

$\therefore$ The general solution of the given equation is

$\mathbf{z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)}$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*iv*) We have $(D^4 + D'^4 - 2D^2D'^2)z = 0$ ...(1)

The auxiliary equation of (1) is $m^4 + 1 - 2m^2 = 0$.

$\therefore \quad (m^2 - 1)^2 = 0 \quad \text{or} \quad m = 1, 1, -1, -1.$

$\therefore$ The general solution of the given equation is

$$\mathbf{z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - x) + x\phi_4(y - x)},$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are arbitrary functions.

(*v*) We have $(D^3D'^2 + D^2D'^3)z = 0$. ...(1)

(1) $\Rightarrow \quad D'^2D^2(D + D')z = 0$

The part of general solution corresponding to the factor D'^2 is $\phi_1(x) + y\phi_2(x)$.

The auxiliary equation of $D^2(D + D')z = 0$ is $m^2(m + 1) = 0$

$\therefore \quad m = 0, 0, -1$

$\therefore$ The part of general solution corresponding to the factor $D^2(D + D')$ is $\phi_3(y + 0.x) + x\phi_4(y + 0.x) + \phi_5(y + (-1)x)$.

$\therefore$ The general solution of the given equation is

$$\mathbf{z = \phi_1(x) + y\phi_2(x) + \phi_3(y) + x\phi_4(y) + \phi_5(y - x)},$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5 are arbitrary functions.

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Express the given equation in terms of D and D′.

Step II. Put D = m and D′ = 1 in the operator of the equation and equate it to zero. This is the auxiliary equation of the given equation. Solve the auxiliary equation.

Step III. If m_i is a distinct root then the corresponding part of the general solution is $\phi(y + m_i x)$. If m_i is a root repeated r times then the corresponding part of the general solution is $\phi_1(y + m_i x) + x\phi_2(y + m_i x) + \ldots\ldots + x^{r-1}\phi_r(y + m_i x)$.

Step IV. If D'^r is a factor of the operator of the equation then we put $D = m$ and $D' = 1$ in the other factor of the operator and the part of the general solution corresponding to D'^r is taken as $\phi_1(x) + y\phi_2(x) + \ldots\ldots + y^{r-1}\phi_r(x)$.

EXERCISE A

Find the general solution of the following partial differential equations :

1. $r = a^2 t$
2. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$
3. $\left(\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial^2 z}{\partial y^2}\right) = 0$
4. $(D^2 - 4DD' + 4D'^2)\, z = 0$
5. $25r - 40s + 16t = 0$
6. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)\, z = 0$
7. $(D^3 - 3DD'^2 + 2D'^3)\, z = 0$
8. $(D^3 + D^2D' - 6DD'^2)\, z = 0$
9. $(D^3 - 2D^2D')\, z = 0$
10. $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x \partial y^2} + 6\dfrac{\partial^3 z}{\partial y^3} = 0$
11. $(D^3 - 3D^2D' + 3DD'^2 - D'^3)\, z = 0$
12. $(D^2D' - 3DD'^2 + 2D'^3)\, z = 0$
13. $(D^4 - 2D^3D' + 2DD'^3 - D'^4)\, z = 0$
14. $(D'^3D + D'^4)\, z = 0$.

Answers

1. $z = \phi_1(y + ax) + \phi_2(y - ax)$
2. $z = \phi_1(y + x) + \phi_2(y - x)$
3. $z = \phi_1(y + (1 + \sqrt{2})x) + \phi_2(y + (1 - \sqrt{2})x)$
4. $z = \phi_1(y + 2x) + x\phi_2(y + 2x)$
5. $z = \phi_1\left(y + \dfrac{4}{5}x\right) + x\phi_2\left(y + \dfrac{4}{5}x\right)$
6. $z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x)$
7. $z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - 2x)$
8. $z = \phi_1(y) + \phi_2(y + 2x) + \phi_3(y - 3x)$
9. $z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$
10. $z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - 3x)$
11. $z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$
12. $z = \phi_1(x) + \phi_2(y + x) + \phi_3(y + 2x)$
13. $z = \phi_1(y - x) + \phi_2(y + x) + x\phi_3(y + x) + x^2\phi_4(y + x)$
14. $z = \phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \phi_4(y - x)$.

GENERAL SOLUTION OF HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION f(D, D′)z = F(x, y) WITH CONSTANT COEFFICIENTS

Let $\quad f(D, D')\, z = F(x, y) \qquad \ldots(1)$

be a homogeneous linear partial differential equation with constant coefficients.

Let u be the general solution of $f(D, D')\, z = 0$.

$\therefore \qquad f(D, D')\, u = 0 \qquad \ldots(2)$

Let v be a particular integral *i.e.,* a particular solution of $f(D, D')\, z = F(x, y)$.

$\therefore \qquad f(D, D')\, v = F(x, y) \qquad \ldots(3)$

Now, $\quad f(D, D')(u+v) = f(D, D')u + f(D, D')v = 0 + F(x, y) = F(x, y)$

(Using (2) and (3))

$\therefore$ $u + v$ is a solution of $f(D, D')z = F(x, y)$. Since u is the general solution of the equation $f(D, D')z = 0$, it contains arbitrary functions equal in number to its order.

$\therefore$ The solution $u + v$ of the equation $f(D, D')z = F(x, y)$ also contains as many arbitrary functions as the order of $f(D, D')z = F(x, y)$.

$\therefore$ $u + v$ is the general solution of the equation $f(D, D')z = F(x, y)$.

The general solution u of the equation $f(D, D')z = 0$ is called the **complementary function (C.F.)** of the equation $f(D, D')z = F(x, y)$.

$\therefore$ **The general solution of the equation f(D, D′)z = F(x, y) is obtained by adding the general solution of the equation f(D, D′)z = 0 to any particular integral of the equation f(D, D′)z = F(x, y).**

PARTICULAR INTEGRAL OF f(D, D′)z = F(x, y)

Let $\quad f(D, D')z = F(x, y) \quad$ ...(1)

be a homogeneous linear partial differential equation with constant coefficients.

The **inverse operator** $\dfrac{1}{f(D, D')}$ of the operator $f(D, D')$ is defined by the identity

$$f(D, D')\left(\frac{1}{f(D, D')}F(x, y)\right) = F(x, y)$$

$\therefore$ $\dfrac{1}{f(D, D')}F(x, y)$ is a particular integral of the equation (1) because

$$f(D, D')\left(\frac{1}{f(D, D')}F(x, y)\right) = F(x, y) = \text{R.H.S. of (1).}$$

$\therefore$ $\mathbf{\dfrac{1}{f(D, D')}}$ **F(x, y) is a particular integral of the equation f(D, D′)z = F(x, y).**

PARTICULAR INTEGRAL WHEN F(X, Y) IS SUM OR DIFFERENCE OF TERMS OF THE FORM $x^m y^n$

If $F(x, y)$ is sum or difference of terms of the form $x^m y^n$, then the particular integral $\dfrac{1}{f(D, D')}F(x, y)$ of the differential equation $f(D, D')z = F(x, y)$ is obtained by expanding $\dfrac{1}{f(D, D')}$ in an infinite series in ascending powers of either D or D′. The particular integrals obtained in the above mentioned two methods may not be identical. Any of the two particular integrals may be used. If $m < n$, then it is advisible to expand $\dfrac{1}{f(D, D')}$ in ascending powers of D and in case $n < m$, then we should expand $\dfrac{1}{f(D, D')}$ in ascending powers of D′.

SOLVED EXAMPLES

Example 2. *Find the general solution of the following partial differential equations*:

(*i*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$ (*ii*) $r + (a + b)s + abt = xy$.

Sol. (*i*) We have $(2D^2 - 5DD' + 2D'^2)\, z = 24(y - x)$. ...(1)

The A.E. of (1) is $2m^2 - 5m + 2 = 0$.

$\therefore \quad m = 1/2,\ 2.$

$$\therefore \quad \text{C.F.} = \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x).$$

Now,

$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}\, 24(y - x)$$

$$= \frac{1}{2D^2}\left(1 - \left(\frac{5D'}{2D} - \frac{D'^2}{D^2}\right)\right)^{-1} 24(y - x)$$

$$= \frac{24}{2D^2}\left(1 + \left(\frac{5D'}{2D} - \frac{D'^2}{D^2}\right) + \ldots\ldots\right)(y - x)$$

$$= \frac{12}{D^2}\left((y - x) + \frac{5}{2D}(1) + 0\right) = 12\left(\frac{x^2 y}{2} - \frac{x^3}{6}\right) + \frac{30}{D^3}(1)$$

$$= 6x^2 y - 2x^3 + 30 \,.\, \frac{x^3}{6} = 6x^2 y + 3x^3.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x) + 6x^2 y + 3x^3},$$

where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) We have $r + (a + b)\, s + abt = xy$.

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} + (a + b)\frac{\partial^2 z}{\partial x \partial y} + ab\frac{\partial^2 z}{\partial y^2} = xy$$

$\Rightarrow \quad (D^2 + (a + b)\, DD' + ab\, D'^2)\, z = xy$ ...(1)

The A.E. of (1) is $m^2 + (a + b)\, m + ab = 0$.

$\therefore \quad m = -a, -b$

$\therefore \quad \text{C.F.} = \phi_1(y - ax) + \phi_2(y - bx).$

Now,

$$\text{P.I.} = \frac{1}{D^2 + (a + b)\, DD' + D'^2}\,(xy)$$

$$= \frac{1}{D^2}\left[1 + \left((a + b)\frac{D'}{D} + \frac{D'^2}{D^2}\right)\right]^{-1}(xy)$$

$$= \frac{1}{D^2}\left(1 - (a + b)\frac{D'}{D} - \frac{D'^2}{D^2} + \ldots\ldots\right)(xy)$$

$$= \frac{1}{D^2}\left[xy - \frac{(a + b)}{D}(x) - 0\right] = y\left(\frac{x^3}{6}\right) - (a + b)\frac{x^4}{24}.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y - ax) + \phi_2(y - bx) + \frac{1}{6}x^3 y - \frac{a + b}{24}x^4},$$

where ϕ_1 and ϕ_2 are arbitrary functions.

NOTES

EXERCISE B

Find the general solution of the following partial differential equations :

1. $(D^2 + 3DD' + 2D'^2)\ z = 2x + 3y$
2. $(D^2 - 2DD' + D'^2)\ z = 12xy$
3. $(D^2 - DD' - 6D'^2)\ z = xy$
4. $(D^2 - a^2D'^2)\ z = x^2$
5. $(D^2 - 6DD' + 9D'^2)\ z = 12x^2 + 36xy$
6. $(D^2 - 2DD' - 15D'^2)z = 12xy.$

Answers

1. $z = \phi_1(y - x) + \phi_2(y - 2x) - \frac{7}{6}x^3 + \frac{3}{2}x^2y$
2. $z = \phi_1(y + x) + x\phi_2(y + x) + 2x^3y + x^4$
3. $z = \phi_1(y - 2x) + \phi_2(y + 3x) + \frac{1}{6}x^3y + \frac{1}{24}x^4$
4. $z = \phi_1(y + ax) + \phi_2(y - ax) + \frac{1}{12}x^4$
5. $z = \phi_1(y + 3x) + x\phi_2(y + 3x) + 10x^4 + 6x^3y$
6. $z = \phi_1(y - 3x) + \phi_2(y + 5x) + x^4 + 2x^3y.$

PARTICULAR INTEGRAL WHEN F(x, y) IS OF THE FORM ϕ(ax + by)

Theorem. If f(D, D′) be a homogeneous function of D and D′ of degree n, then the particular integral of f(D, D′)z = ϕ(ax + by) is given by

$$\frac{1}{f(a, b)} \iint \ldots\ldots \int \phi(v)\, dv\, dv \ldots\ldots dv, \text{ where } v = ax + by, \text{ provided } f(a, b) \neq 0.$$

Proof. We have
$$D^r\phi(ax + by) = a^r\phi^{(r)}(ax + by),$$
$$D'^s\phi(ax + by) = b^s\phi^{(s)}(ax + by)$$
and
$$D^rD'^s\phi(ax + by) = a^rb^s\phi^{(r+s)}(ax + by).$$
$$\therefore \quad f(D, D')\ \phi(ax + by) = f(a, b)\ \phi^{(n)}\ (ax + by)$$
($\because$ f(D, D′) is a homogeneous function of degree n)

Dividing both sides by non-zero constant $f(a, b)$, we get
$$f(D, D')\ \frac{\phi(ax + by)}{f(a, b)} = \phi^{(n)}(ax + by)$$

$\therefore$ By definition of the inverse operator f(D, D′), we have
$$\frac{1}{f(D, D')}\ \phi^{(n)}(ax + by) = \frac{\phi(ax + by)}{f(a, b)}$$
$$\Rightarrow \quad \frac{1}{f(D, D')}\ \phi^{(n)}(v) = \frac{1}{f(a, b)}\ \phi(v), \text{ where } v = ax + by.$$

Replacing $\phi^{(n)}(v)$ by $\phi(v)$, we get
$$\frac{1}{f(D, D')}\ \phi(v) = \frac{1}{f(a, b)} \iint \ldots\ldots \int \phi(v)\, dv\, dv \ldots\ldots dv, \quad \text{where } v = ax + by.$$

On the right side, the function $\phi(v)$ is to be integrated n times w.r.t. v, which is also the degree of the homogeneous function f(D, D′).

$$\therefore \quad \textbf{P.I.} = \frac{1}{f(D, D')}\ \phi(ax + by) = \frac{1}{f(a, b)} \iint \ldots\ldots \int \phi(v)\, dv\, dv \ldots\ldots dv,$$

where v = ax + by, provided f(a, b) ≠ 0.

Exceptional Case. If $f(a, b) = 0$, then $bD - aD'$ must be a factor of $f(D, D')$ because $ba - ab = 0$. Let the factor $bD - aD'$ be repeated r times, where $r \geq 1$. The value of

$\frac{1}{(bD - aD')^r}\,\phi(ax + by)$ is given by $\frac{x^r}{b^r r!}\,\phi(ax + by)$.

$$\therefore \quad \mathbf{\frac{1}{(bD - aD')^r}\,\phi(ax + by) = \frac{x^r}{b^r r!}\,\phi(ax + by).}$$

SOLVED EXAMPLES

Example 3. *Find the general solution of the following partial differential equations :*

(*i*) $(2D^2 - 5DD' + 2D'^2)\,z = 5(y - x)$

(*ii*) $r + s - 2t = (2x + y)^{1/2}$

(*iii*) $(D^3 - 4D^2D' + 4DD'^2)\,z = 6 \sin (3x + 2y)$

(*iv*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\,z = e^{5x+6y}$

Sol. (*i*) We have $\quad (2D^2 - 5DD' + 2D'^2)z = 5(y - x)$. ...(1)

The A.E. of (1) is $2m^2 - 5m + 2 = 0$.

$$\therefore \quad m = 1/2,\ 2$$

$$\therefore \quad \text{C.F.} = \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x).$$

$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}\,5(y - x) \quad ...(2)$$

$$y - x = ax + by \ \Rightarrow\ a = -1,\ b = 1.$$

Also $\quad f(D, D') = 2D^2 - 5DD' + 2D'^2$.

$$\therefore \quad f(a, b) = f(-1, 1) = 2(-1)^2 - 5(-1)(1) + 2(1)^2 = 9 \neq 0$$

$$\therefore \quad (2) \Rightarrow \quad \text{P.I.} = \frac{1}{f(a,b)}\iint 5v\,dv\,dv, \text{ where } v = y - x$$

$$= \frac{1}{9}.5.\int \frac{v^2}{2}\,dv = \frac{5}{9}.\frac{v^3}{6} = \frac{5}{54}(y - x)^3 .$$

Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1\left(y + \frac{1}{2}x\right) + \phi_2(y + 2x) + \frac{5}{54}(y - x)^3},$$

where ϕ_1, ϕ_2 are arbitrary functions.

(*ii*) We have $\quad r + s - 2t = (2x + y)^{1/2}$.

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (2x + y)^{1/2}$$

$$\Rightarrow \quad (D^2 + DD' - 2D'^2)\,z = (2x + y)^{1/2} \quad ...(1)$$

The A.E. of (1) is $\quad m^2 + m - 2 = 0$.

$$\therefore \quad m = -2,\ 1$$

$$\therefore \quad \text{C.F.} = \phi_1(y - 2x) + \phi_2(y + x).$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2}\,(2x + y)^{1/2} \quad ...(2)$$

$$2x + y = ax + by \ \Rightarrow\ a = 2,\ b = 1.$$

Also, $f(D, D') = D^2 + DD' - 2D'^2$.

$\therefore$ $f(a, b) = f(2, 1) = (2)^2 + (2)(1) - 2(1)^2 = 4 \neq 0$.

$\therefore$ (2) $\Rightarrow$ $\text{P.I.} = \frac{1}{f(a,b)} \iint v^{1/2}\, dv\, dv$, where $v = 2x + y$

$$= \frac{1}{4} \int \frac{2v^{3/2}}{3} dv = \frac{1}{4} \cdot \frac{2}{3} \frac{v^{5/2}}{5/2} = \frac{1}{15} (2x + y)^{5/2}.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \phi_1(y - 2x) + \phi_2(y + x) + \frac{1}{15}(2x + y)^{5/2}}$, where ϕ_1, ϕ_2 are arbitrary functions.

(*iii*) We have $(D^3 - 4D^2D' + 4DD'^2)\, z = 6 \sin (3x + 2y)$. ...(1)

The A.E. of (1) is $m^3 - 4m^2 + 4m = 0$.

$\therefore$ $m = 0, 2, 2$

$\therefore$ $\text{C.F.} = \phi_1(y + 0.x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

$= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 6 \sin (3x + 2y) \quad ...(2)$$

$3x + 2y = ax + by \Rightarrow a = 3, b = 2$.

Also, $f(D, D') = D^3 - 4D^2D' + 4DD'^2$.

$\therefore$ $f(a, b) = f(3, 2) = (3)^3 - 4(3)^2(2) + 4(3)(2)^2 = 3 \neq 0$.

$\therefore$ (2) $\Rightarrow$ $\text{P.I.} = \frac{1}{f(a,b)} \iiint 6 \sin v\, dv\, dv\, dv$, where $v = 3x + 2y$

$$= \frac{1}{3} . 6 \iint - \cos v\, dv\, dv = 2 \int (- \sin v)\, dv$$

$= 2 \cos v = 2 \cos (3x + 2y)$.

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + 2 \cos (3x + 2y),}$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*iv*) We have $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\, z = e^{5x + 6y}$. ...(1)

The A.E. of (1) is $m^3 - 6m^2 + 11m - 6 = 0$

$\therefore$ $m = 1, 2, 3$

$\therefore$ $\text{C.F.} = \phi_1(y + 1\,.\,x) + \phi_2(y + 2x) + \phi_3(y + 3x)$.

$$\text{P.I.} = \frac{1}{D^3 - 6D^2D' + 11DD'^2 - 6D'^3} e^{5x+6y} \quad ...(2)$$

$5x + 6y = ax + by \Rightarrow a = 5, b = 6$.

Also, $f(D, D') = D^3 - 6D^2D' + 11DD'^2 - 6D'^3$

$\therefore$ $f(a, b) = f(5, 6) = (5)^3 - 6(5)^2(6) + 11(5)(6)^2 - 6(6)^3 = - 91 \neq 0$.

$\therefore$ (2) $\Rightarrow$ $\text{P.I.} = \frac{1}{f(a,b)} \iiint e^v\, dv\, dv\, dv$, where $v = 5x + 6y$

$$= \frac{1}{-91} \iint e^v\, dv\, dv = -\frac{1}{91} \int e^v\, dv = -\frac{1}{91} e^v$$

$$= -\frac{1}{91} e^{5x+6y}.$$

$\therefore$ Using G.S. = C.F. +P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x) - \frac{1}{91} e^{5x+6y}},$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Example 4. *Find the general solution of the following partial differential equations :*

(*i*) $(D^3 - 4D^2D' + 4DD'^2)z = 4 \sin (2x + y)$

(*ii*) $(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos (x - y)$

(*iii*) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)\, z = e^{y+2x} + (y + x)^{1/2}$

(*iv*) $(D^3 - 7DD'^2 - 6D'^3)z = \sin (x + 2y) + e^{3x+y}$.

Sol. (*i*) We have $(D^3 - 4D^2D' + 4DD'^2)\, z = 4 \sin (2x + y)$. ...(1)

The A.E. of (1) is $m^3 - 4m^2 + 4m = 0$.

$\therefore \qquad m = 0, 2, 2$

$\therefore \qquad \text{C.F.} = \phi_1(y + 0.x) + \phi_2(y + 2x) + x\phi_3(y + 2x)$.

$$\text{P.I.} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} 4 \sin (2x + y) \qquad ...(2)$$

$2x + y = ax + by \;\Rightarrow\; a = 2, b = 1.$

Also, $\qquad f(D, D') = D^3 - 4D^2D' + 4DD'^2$.

$\therefore \qquad f(a, b) = f(2, 1) = (2)^3 - 4(2)^2(1) + 4(2)(1)^2 = 0.$

$D^3 - 4D^2D' + 4DD'^2 = D(D^2 - 4DD' + 4D'^2) = D(D - 2D')^2$

$\therefore$ (2) $\Rightarrow$ $\text{P.I.} = \dfrac{1}{(D - 2D')^2} \cdot \dfrac{1}{D} 4 \sin (2x + y)$ $(\because D = 2, D' = 1 \Rightarrow D - 2D' = 0)$

$$= \frac{1}{(D - 2D')^2} \int 4 \sin (2x + y)\, dx^*$$

$$= \frac{1}{(D - 2D')^2} \cdot \frac{4(-\cos (2x + y))}{2}$$

$$= -2 \frac{1}{(D - 2D')^2} \cos (2x + y)$$

$$= -2 \,.\, \frac{x^2}{(1)^2\, 2!} \cos (2x + y) \quad (\because\; bD - aD' = (1)D - 2D' = D - 2D')$$

$$= -x^2 \cos (2x + y).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is $\mathbf{z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - x^2 \cos (2x + y)}$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*ii*) We have $(D^3 - 7DD'^2 - 6D'^3)\, z = x^2 + xy^2 + y^3 + \cos (x - y)$. ...(1)

The A.E. of (1) is $m^3 - 7m - 6 = 0$.

$\Rightarrow \qquad m = -1, -2, 3$

$\therefore \quad \text{C.F.} = \phi_1(y + (-1)x) + \phi_2(y + (-2)x) + \phi_3(y + 3x)$.

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*Alternatively, $\dfrac{1}{D} 4 \sin (2x + y) = \dfrac{1}{2} \displaystyle\int 4 \sin v\, dv$, where $v = 2x + y$

$= -2 \cos v = -2 \cos (2x + y)$.

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3 + \cos(x-y))$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}\cos(x-y).$$

Now, $\frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3)$

$$= \frac{1}{D^3}\left[1 - \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right)\right]^{-1}(x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}\left[1 + \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right) + ...\right](x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{7}{D^5}(2x + 6y) + \frac{6}{D^6}(6)$$

$$= \left(\frac{x^5}{60} + \frac{x^4y^2}{24} + \frac{x^3y^3}{6}\right) + 7\left(\frac{x^6}{360} + \frac{x^5y}{20}\right) + 36\left(\frac{x^6}{720}\right)$$

$$= \frac{5}{72}x^6 + \frac{1}{60}x^5 + \frac{7}{20}x^5y + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3.$$

Also $\frac{1}{D^3 - 7DD'^2 - 6D'^3}\cos(x-y)$

$$= \frac{1}{(D+D')(D+2D')(D-3D')}\cos(x-y)$$

$$= \frac{1}{D+D'}\cdot\frac{1}{(D+2D')(D-3D')}\cos(x-y)$$

$(\because\ D = 1,\ D' = -1 \Rightarrow D + D' = 0)$

$$= \frac{1}{D+D'}\cdot\frac{1}{(1+2(-1))(1-3(-1))}\iint \cos v\, dv\, dv, \text{ where } v = x - y$$

$$= \frac{1}{D+D'}\cdot\frac{1}{-4}(-\cos v) = \frac{1}{D+D'}\cdot\left(\frac{1}{4}\cos(x-y)\right)$$

$$= \frac{1}{4}\cdot\frac{1}{D+D'}\cos(x-y) = -\frac{1}{4}\cdot\frac{1}{(-D-D')}\cos(x-y) \quad \text{(Note this step)}$$

$$= -\frac{1}{4}\cdot\frac{x'}{(-1)^1\,1!}\cos(x-y) \qquad (\because\ bD - aD' = (-1)D - (1)D' = -D - D')$$

$$= \frac{1}{4}x\cos(x-y)$$

$\therefore$ $$\text{P.I.} = \frac{5}{72}x^6 + \frac{1}{60}x^5 + \frac{7}{20}x^5y + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3 + \frac{x}{4}\cos(x-y).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \frac{5}{72}x^6 + \frac{1}{60}x^5 + \frac{7}{20}x^5y + \frac{1}{24}x^4y^2 + \frac{1}{6}x^3y^3}$$

$\mathbf{+ \frac{x}{4}\cos(x - y)}$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*iii*) We have $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)\, z = e^{y+2x} + (y + x)^{1/2}$. ...(1)

The A.E. of (1) is $m^3 - 4m^2 + 5m - 2 = 0$.

$\therefore \quad m = 1, 1, 2$

$\therefore \quad$ C.F. $= \phi_1(y + 1 \,.\, x) + x\phi_2(y + 1 \,.\, x) + \phi_3(y + 2x)$.

$\therefore \quad$ P.I. $= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3}(e^{y+2x} + (y + x)^{1/2})$

$$= \frac{1}{(D - D')^2\,(D - 2D')} e^{y+2x} + \frac{1}{(D - D')^2\,(D - 2D')}(y + x)^{1/2}.$$

Now, $\dfrac{1}{(D - D')^2\,(D - 2D')} e^{y+2x} = \dfrac{1}{D - 2D'} \cdot \dfrac{1}{(D - D')^2} e^{2x+y}$

($\because$ D = 2, D′ = 1 $\Rightarrow$ D – 2D′ = 0)

$$= \frac{1}{D - 2D'} \cdot \frac{1}{(2 - 1)^2} \iint e^v\, dv\, dv, \text{ where } v = 2x + y$$

$$= \frac{1}{D - 2D'} e^{2x+y} = \frac{x^1}{(1)^1\, 1!} e^{2x+y} = xe^{2x+y}.$$

($\because$ bD – aD′ = (1)D – 2D′ = D – 2D′)

Also $\dfrac{1}{(D - D')^2 (D - 2D')}(y + x)^{1/2} = \dfrac{1}{(D - D')^2} \cdot \dfrac{1}{D - 2D'}(x + y)^{1/2}$

($\because$ D = 1, D′ = 1 $\Rightarrow$ D – D′ = 0)

$$= \frac{1}{(D - D')^2} \cdot \frac{1}{1 - 2(1)} \int v^{1/2} dv, \text{ where } v = x + y$$

$$= \frac{1}{(D - D')^2} \cdot -\frac{v^{3/2}}{3/2} = -\frac{2}{3} \cdot \frac{1}{(D - D')^2}(x + y)^{3/2}$$

$$= -\frac{2}{3} \cdot \frac{x^2}{(1)^2\, 2!}(x + y)^{3/2}$$ ($\because$ bD – aD′ = (1)D – (1)D′ = D – D′)

$$= -\frac{1}{3} x^2 (x + y)^{3/2}$$

$\therefore \quad$ P.I. $= xe^{2x+y} - \dfrac{1}{3} x^2 (x + y)^{3/2}$.

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x) + xe^{2x+y} - \frac{1}{3} x^2\, (x + y)^{3/2}},$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*iv*) We have $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x + 2y) + e^{3x+y}$. ...(1)

The A.E. of (1) is $m^3 - 7m - 6 = 0$.

$\therefore \quad m = -1, -2, 3$

$\therefore \quad$ C.F. $= \phi_1(y + (-1)x) + \phi_2(y + (-2)x) + \phi_3(y + 3x)$.

P.I. $= \dfrac{1}{D^3 - 7DD'^2 - 6D'^3}(\sin(x + 2y) + e^{3x+y})$

$$= \frac{1}{(D + D')(D + 2D')(D - 3D')} \sin(x + 2y) + \frac{1}{(D + D')(D + 2D')(D - 3D')} e^{3x+y}.$$

NOTES

Now, $\frac{1}{(D+D')(D+2D')(D-3D')}\sin(x+2y)$

$$= \frac{1}{(1+2)(1+4)(1-6)}\iiint \sin v\,dv\,dv\,dv, \text{ where } v = x+2y$$

$$= -\frac{1}{75}\iint -\cos v\,dv\,dv = \frac{1}{75}\int \sin v\,dv = \frac{1}{75}(-\cos v) = -\frac{1}{75}\cos(x+2y).$$

Also $\frac{1}{(D+D')(D+2D')(D-3D')}e^{3x+y}$

$$= \frac{1}{D-3D'}\cdot\frac{1}{(D+D')(D+2D')}e^{3x+y}$$

$(\because \;\; D = 3,\ D' = 1 \;\;\Rightarrow\;\; D - 3D' = 0)$

$$= \frac{1}{D-3D'}\cdot\frac{1}{(3+1)(3+2)}\iint e^{v}\,dv\,dv, \text{ where } v = 3x+y$$

$$= \frac{1}{D-3D'}\cdot\frac{1}{20}e^{3x+y} = \frac{x^1}{(1)^1\,1!}\cdot\frac{e^{3x+y}}{20}$$

$(\because \;\; bD - aD' = (1)D - 3D' = D - 3D')$

$$= \frac{1}{20}xe^{3x+y}$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{1}{20}xe^{3x+y}},$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

EXERCISE C

Find the general solution of the following partial differential equations :

1. $(D^2 + 3DD' + 2D'^2)\,z = 2x + 3y$
2. $(D^2 + 2DD' + D'^2)\,z = e^{2x+3y}$
3. $(D^2 - 2DD' + D'^2)\,z = e^{x+2y}$
4. $(D^3 - 4D^2D' + 4DD'^2)\,z = \cos(2x+3y)$
5. $s = e^{x+y}$
6. $4r - 4s + t = 16\log(x+2y)$
7. $2r - s - 3t = 5e^x/e^y$
8. $(D^2 - 5DD' + 4D'^2)\,z = \sin(4x+y)$
9. $(D^2 - 2aDD' + a^2D'^2)\,z = g(y+ax)$
10. $(2D^2 - 5DD' + 2D'^2)\,z = 5\sin(2x+y)$
11. $(D^3 - 2D^2D' - DD'^2 + 2D'^3)\,z = e^{x+y}$
12. $(D^3 - 4D^2D' + 4DD'^2)\,z = \sin(y+2x)$
13. $(D^3 - 4D^2D' + 4DD'^2)\,z = \cos(2x+y)$
14. $(D^2 - 3DD' + 2D'^2)\,z = e^{2x-y} + e^{x+y} + \cos(x + 2y)$.

Answers

1. $z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{240}(2x+3y)^3$
2. $z = \phi_1(y-x) + x\phi_2(y-x) + \frac{e^{2x+3y}}{25}$
3. $z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y}$
4. $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - \frac{1}{32}\sin(2x+3y)$
5. $z = \phi_1(x) + \phi_2(y) + e^{x+y}$
6. $z = \phi_1\left(y + \frac{1}{2}x\right) + x\phi_2\left(y + \frac{1}{2}x\right) + 2x^2\log(x+2y)$
7. $z = \phi_1(y-x) + \phi_2\left(y + \frac{3}{2}x\right) + xe^{x-y}$
8. $z = \phi_1(y+x) + \phi_2(y+4x) - \frac{1}{3}x\cos(4x+y)$

9. $z = \phi_1(y + ax) + x\phi_2(y + ax) + \dfrac{x^2}{2} g(y + ax)$

10. $z = \phi_1\left(y + \dfrac{1}{2}x\right) + \phi_2(y + 2x) - \dfrac{5}{3} x \cos(2x + y)$

11. $z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + 2x) - \dfrac{1}{2} xe^{x+y}$

12. $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - \dfrac{x^2}{4} \cos(2x + y)$

13. $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + \dfrac{x^2}{4} \sin(2x + y)$

14. $z = \phi_1(y + x) + \phi_2(y + 2x) + \dfrac{1}{12} e^{2x-y} - xe^{x+y} - \dfrac{1}{3} \cos(x + 2y)$.

GENERAL METHOD OF FINDING PARTICULAR INTEGRAL

Let $$f(D, D')z = F(x, y) \qquad ...(1)$$
be a homogeneous linear partial differential equation of order n with constant coefficients.

Let $m_1, m_2, ..., m_n$ be the roots of the auxiliary equation of (1). Here some of the roots may be repeated.

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$
$$= \frac{1}{(D - m_1D')(D - m_2D')......(D - m_nD')} F(x, y)$$

We first study the method of evaluating $\dfrac{1}{D - mD'} g(x, y)$ for some constant m and function $g(x, y)$.

Let $$z = \frac{1}{D - mD'} g(x, y)$$

$\therefore$ $(D - mD')\, z = g(x, y)$

$\Rightarrow$ $p - mq = g(x, y)$

$\therefore$ Lagrange auxiliary equations are $$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{g(x, y)} \qquad ...(2)$$

(2) $\Rightarrow$ $dx = \dfrac{dy}{-m}$ $\Rightarrow$ $dy + m\,dx = 0$ $\Rightarrow$ $y + mx = a$

(2) $\Rightarrow$ $dx = \dfrac{dz}{g(x, y)}$

$\Rightarrow$ $dz = g(x, a - mx)\, dx$ $\Rightarrow$ $z = \int g(x, a - mx)\, dx$

$\therefore$ $$\mathbf{\frac{1}{D - mD'} g(x, y) = \int g(x, a - mx)\, dx} \qquad ...(3)$$

After evaluation of right side, the constant a is replaced by $y + mx$.

Now we come to the evaluation of a particular integral of (1).

$$\text{P.I. of (1)} = \frac{1}{(D - m_1D')(D - m_2D')......(D - m_nD')} F(x, y)$$

$$= \frac{1}{D - m_1D'} \cdot \frac{1}{D - m_2D'} ...... \frac{1}{D - m_nD'} F(x, y)$$

This right side is evaluated by repeated application of the formula (3).

SOLVED EXAMPLES

Example 5. *Find the general solution of the following partial differential equations:*

(*i*) $(D^2 + 2DD' + D'^2)\, z = 2 \cos y - x \sin y$

(*ii*) $\dfrac{\partial^2 z}{\partial x^2} - 4 \dfrac{\partial^2 z}{\partial y^2} = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$

(*iii*) $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = \tan^3 x \tan y - \tan x \tan^3 y$

(*iv*) $(D^2 + DD' - 6D'^2)\, z = x^2 \sin (x + y)$.

Sol. (*i*) We have $(D^2 + 2DD' + D'^2)\, z = 2 \cos y - x \sin y$. ...(1)

The A.E. of (1) is $m^2 + 2m + 1 = 0$.

$\therefore$ $m = -1, -1$

$\therefore$ $\text{C.F.} = \phi_1(y + (-1)x) + x\phi_2(y + (-1)x) = \phi_1(y - x) + x\phi_2(y - x)$.

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2} (2 \cos y - x \sin y)$$

$$= \frac{1}{(D + D')(D + D')} (2 \cos y - x \sin y)$$

$$= \frac{1}{D + D'} \int [2 \cos (a + x) - x \sin (a + x)]\, dx$$

$(D + D' = D - mD' \Rightarrow m = -1. \quad \therefore \quad y = a - mx = a + x)$

$$= \frac{1}{D + D'} [2 \sin (a + x) - \{- x \cos (a + x) + \sin (a + x)\}]$$

$$= \frac{1}{D + D'} [\sin (a + x) + x \cos (a + x)] = \frac{1}{D + D'} [\sin y + x \cos y]$$

$$= \int (\sin (a + x) + x \cos (a + x))\, dx$$

$(D + D' = D - mD' \Rightarrow m = -1. \quad \therefore \quad y = a - mx = a + x)$

$= - \cos (a + x) + x \sin (a + x) + \cos (a + x)$

$= x \sin (a + x) = x \sin y$.

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \phi_1(y - x) + x\phi_2(y - x) + x \sin y}$, where ϕ_1, ϕ_2 are arbitrary constants.

(*ii*) We have $\dfrac{\partial^2 z}{\partial x^2} - 4 \dfrac{\partial^2 z}{\partial y^2} = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$.

$\Rightarrow$ $(D^2 - 4D'^2)\, z = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$ ...(1)

The A.E. of (1) is $m^2 - 4 = 0$.

$\therefore$ $m = -2, 2$

$\therefore$ $\text{C.F.} = \phi_1(y + (-2)x) + \phi_2(y + 2x) = \phi_1(y - 2x) + \phi_2(y + 2x)$.

$$\text{P.I.} = \frac{1}{D^2 - 4D'^2}\left(\frac{4x}{y^2} - \frac{y}{x^2}\right) = \frac{1}{(D+2D')(D-2D')}\left(\frac{4x}{y^2} - \frac{y}{x^2}\right)$$

$$= \frac{1}{D+2D'}\int\left(\frac{4x}{(a-2x)^2} - \frac{a-2x}{x^2}\right)dx$$

$$(D - 2D' = D - mD' \quad\Rightarrow\quad m = 2 \quad\therefore\quad y = a - mx = a - 2x)$$

$$= \frac{1}{D+2D'}\int\left(-\frac{2a-4x-2a}{(a-2x)^2} - \frac{a}{x^2} + \frac{2}{x}\right)dx$$

$$= \frac{1}{D+2D'}\int\left(-\frac{2}{a-2x} + \frac{2a}{(a-2x)^2} - \frac{a}{x^2} + \frac{2}{x}\right)dx$$

$$= \frac{1}{D+2D'}\left[\log(a-2x) + \frac{a}{a-2x} + \frac{a}{x} + 2\log x\right]$$

$$= \frac{1}{D+2D'}\left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + 2\log x\right]$$

$$= \frac{1}{D+2D'}\left[\log y + 2\log x + \frac{2x}{y} + \frac{y}{x} + 3\right]$$

$$= \int\left(\log(a+2x) + 2\log x + \frac{2x}{a+2x} + \frac{a+2x}{x} + 3\right)dx$$

$$(D + 2D' = D - mD' \quad\Rightarrow\quad m = -2. \quad\therefore\quad y = a - mx = a + 2x)$$

$$= \int\left(\log(a+2x) + 2\log x + \frac{2x}{a+2x} + \frac{a}{x} + 5\right)dx$$

$$= \log(a+2x).x - \int\frac{2}{a+2x}.x\,dx + 2(\log x).x - \int\frac{2}{x}.x\,dx$$

$$+ \int\frac{2x}{a+2x}dx + a\log x + 5x$$

$$= x\log(a+2x) + (2x+a)\log x + 3x = x\log y + y\log x + 3x.$$

Using G.S. = C.F. + P.I., the general solution of the given equation is

$z = \phi_1(y - 2x) + \phi_2(y + 2x) + x \log y + y \log x + 3x$, where ϕ_1, ϕ_2 are arbitrary functions.

(*iii*) We have $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = \tan^3 x \tan y - \tan x \tan^3 y.$

$\Rightarrow$ $(D^2 - D'^2)\, z = \tan^3 x \tan y - \tan x \tan^3 y$ ...(1)

The A.E. of (1) is $m^2 - 1 = 0$.

$\therefore$ $m = -1, 1$

$\therefore$ C.F. $= \phi_1(y + (-1)x) + \phi_2(y + 1.x) = \phi_1(y - x) + \phi_2(y + x).$

$$\text{P.I.} = \frac{1}{D^2 - D'^2}(\tan^3 x \tan y - \tan x \tan^3 y)$$

$$= \frac{1}{(D+D')(D-D')}(\tan^3 x \tan y - \tan x \tan^3 y)$$

NOTES

$$= \frac{1}{D+D'} \int [\tan^3 x \tan(a-x) - \tan x \tan^3(a-x)]\, dx$$

$$(D - D' = D - mD' \quad \Rightarrow \quad m = 1. \quad \therefore \quad y = a - mx = a - x)$$

$$= \frac{1}{D+D'} \int \tan x \tan(a-x)\,[\sec^2 x - 1 - \sec^2(a-x) + 1]\, dx$$

$$= \frac{1}{D+D'} \left[\int \tan(a-x) \,.\, \tan x \sec^2 x\, dx - \int \tan x \,.\, \tan(a-x) \sec^2(a-x) \right] dx$$

$$= \frac{1}{D+D'} \left[\tan(a-x) \,.\, \frac{\tan^2 x}{2} + \frac{1}{2} \int \sec^2(a-x) \tan^2 x\, dx + \tan x \,.\, \frac{\tan^2(a-x)}{2} - \frac{1}{2} \int \sec^2 x \tan^2(a-x)\, dx \right]$$

$$= \frac{1}{2(D+D')} \left[\tan(a-x) \tan^2 x + \tan x \tan^2(a-x) - \int (\sec^2 x (\sec^2(a-x) - 1) - \sec^2(a-x)(\sec^2 x - 1)\, dx \right]$$

$$= \frac{1}{2(D+D')} \left[\tan(a-x) \tan^2 x + \tan x \tan^2(a-x) - \int (\sec^2(a-x) - \sec^2 x)\, dx \right]$$

$$= \frac{1}{2(D+D')} [\tan y \tan^2 x + \tan x \tan^2 y + \tan y + \tan x]$$

$$= \frac{1}{2(D+D')} [\tan y \sec^2 x + \tan x \sec^2 y]$$

$$= \frac{1}{2} \int [\tan(a+x) \sec^2 x + \tan x \sec^2(a+x)]\, dx$$

$$(D + D' = D - mD' \quad \Rightarrow \quad m = -1. \quad \therefore \quad y = a - mx = a + x)$$

$$= \frac{1}{2} \left[\tan(a+x) \tan x - \int \sec^2(a+x) \tan x\, dx + \int \tan x \sec^2(a+x)\, dx \right]$$

$$= \frac{1}{2} \tan y \tan x.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \phi_1(y-x) + \phi_2(y+x) + \frac{1}{2} \tan x \tan y,}$ where ϕ_1, ϕ_2 are arbitrary functions.

(*iv*) We have $(D^2 + DD' - 6D'^2)\, z = x^2 \sin(x+y)$. ...(1)

The A.E. of (1) is $m^2 + m - 6 = 0$.

$\therefore$ $m = -3, 2$

$\therefore$ C.F. $= \phi_1(y + (-3)x) + \phi_2(y + 2x) = \phi_1(y - 3x) + \phi_2(y + 2x)$.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\, x^2 \sin(x+y)$$

$$= \frac{1}{(D+3D')(D-2D')}\, x^2 \sin(x+y)$$

$$= \frac{1}{D+3D'} \int x^2 \sin(x + a - 2x)\, dx$$

$$(D - 2D' = D - mD' \;\Rightarrow\; m = 2. \quad \therefore \quad y = a - mx = a - 2x)$$

$$= \frac{1}{D+3D'} \int x^2 \sin(a - x)\, dx$$

$$= \frac{1}{D+3D'} \left[x^2 \cos(a - x) - \int 2x \cos(a - x)\, dx \right]$$

$$= \frac{1}{D+3D'} \left[x^2 \cos(a - x) - \left(-2x \sin(a - x) + \int 2 \sin(a - x)\, dx\right) \right]$$

$$= \frac{1}{D+3D'} \left[x^2 \cos(a - x) + 2x \sin(a - x) - 2 \cos(a - x) \right]$$

$$= \frac{1}{D+3D'} \left[(x^2 - 2) \cos(2x + y - x) + 2x \sin(2x + y - x) \right]$$

$$= \frac{1}{D+3D'} \left[(x^2 - 2) \cos(x + y) + 2x \sin(x + y) \right]$$

$$= \int [(x^2 - 2) \cos(x + a + 3x) + 2x \sin(x + a + 3x)]\, dx$$

$$(D + 3D' = D - mD' \;\Rightarrow\; m = -3. \quad \therefore \quad y = a - mx = a + 3x)$$

$$= \int [(x^2 - 2) \cos(4x + a) + 2x \sin(4x + a)]\, dx$$

$$= (x^2 - 2) \frac{\sin(4x + a)}{4} - \int 2x \,.\, \frac{\sin(4x + a)}{4}\, dx + \int 2x \sin(4x + a)\, dx$$

$$= \frac{1}{4}(x^2 - 2) \sin(4x + a) + \frac{3}{2} \int x \sin(4x + a)\, dx$$

$$= \frac{1}{4}(x^2 - 2) \sin(4x + a) + \frac{3}{2} \left[-x \frac{\cos(4x + a)}{4} - \int 1 \,.\, -\frac{\cos(4x + a)}{4}\, dx \right]$$

$$= \frac{1}{4}(x^2 - 2) \sin(4x + a) - \frac{3}{8} x \cos(4x + a) + \frac{3}{32} \sin(4x + a)$$

$$= \left(\frac{x^2}{4} - \frac{13}{32} \right) \sin(4x + y - 3x) - \frac{3}{8} x \cos(4x + y - 3x)$$

$$= \left(\frac{x^2}{4} - \frac{13}{32} \right) \sin(x + y) - \frac{3}{8} x \cos(x + y).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = \phi_1(y - 3x) + \phi_2(y + 2x) + \left(\frac{x^2}{4} - \frac{13}{32} \right) \sin(x + y) - \frac{3}{8} x \cos(x + y),}$$

where ϕ_1, ϕ_2 are arbitrary functions.

EXERCISE D

Find the general solution of the following partial differential equations :

1. $r + s - 6t = y \cos x$

2. $(D^2 + DD' - 6D'^2)\, z = y \sin x$

3. $r - s - 2t = (2x^2 + xy - y^2) \sin xy - \cos xy$

4. $(D^2 - DD' - 2D'^2)\, z = (y - 1)\, e^x$

5. $(D^3 - 3DD'^2 - 2D'^3)\, z = \cos(x + 2y) - e^y (3 + 2x)$

6. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)\, z = (y + 2)\, e^x.$

NOTES

Answers

1. $z = \phi_1(y + 2x) + \phi_2(y - 3x) - y \cos x + \sin x$

2. $z = \phi_1(y + 2x) + \phi_2(y - 3x) - y \sin x - \cos x$

3. $z = \phi_1(y + 2x) + \phi_2(y - x) + \sin xy$

4. $z = \phi_1(y + 2x) + \phi_2(y - x) + ye^x$

5. $z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + 2x) + \dfrac{1}{27} \sin (x + 2y) + xe^y$

6. $z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x) + ye^x$.

5. NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

STRUCTURE

INTRODUCTION

From the last chapter, we have been solving linear partial differential equations with constant coefficients. In that chapter we found the general solution of only such equations in which the orders of all partial derivatives involved in the equation were same. In other words, we solved only homogeneous linear partial differential equations with constant coefficients. In the present chapter, we shall learn the methods of finding general solution of linear partial differential equations which are not homogeneous.

NOTES

NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A linear partial differential equation with constant coefficients is called a **non-homogeneous linear partial differential equation with constant coefficients** if the orders of partial derivatives occurring in the equation are not equal.

For example, the following partial differential equations are all non-homogeneous linear partial differential equations with constant coefficients:

(*i*) $\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2} = e^{2x+3y}$

(*ii*) $(2DD' + D'^2 - 3D')z = 3 \cos (3x - 2y)$

(*iii*) $(D - 2D' + 5)(D^2 + D' + 3)z = e^{3x-4y} \sin (x - 2y)$

REDUCIBLE AND IRREDUCIBLE NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Let $f(D, D')\, z = F(x, y)$ be a non-homogeneous linear partial differential equation with constant coefficients. This equation is called **reducible** if $f(D, D')$ can be resolved into factors each of which is of the first degree in D and D′.

For example, $(D^2 - D'^2 + 3D - 3D')z = \sin x$ is a reducible non-homogeneous linear partial differential equation with constant coefficients because $D^2 - D'^2 + 3D - 3D' = (D - D')\,(D + D' + 3)$. A non-homogeneous linear partial differential equation with constant coefficients is called **irreducible** if it is not reducible. For example $(2D^2 - D'^2 + D)\, z = x^2 - y$ is an irreducible non-homogeneous linear partial differential equation with constant coefficients because $(2D^2 - D'^2 + D)$ cannot be resolved into linear factors in D and D′.

GENERAL SOLUTION OF REDUCIBLE NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION f(D, D′)z = 0 WITH CONSTANT COEFFICIENTS

Let $\quad f(D, D')\, z = (a_1D + b_1D' + c_1) \ldots\ldots (a_nD + b_nD' + c_n)\, z = 0 \quad \ldots(1)$

be a reducible non-homogeneous linear partial differential equation with constant coefficients.

The factors on the left side of (1) may or may not be distinct.

Case I. Factors are distinct

Equation (1) shows that for $1 \le i \le n$, the solution of $(a_iD + b_iD' + c_i)\, z = 0$ is a solution of (1).

$$(a_iD + b_iD' + c_i)\, z = 0 \quad \Rightarrow \quad a_ip + b_iq = -c_iz$$

$$\Rightarrow \quad \frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_iz} \quad \ldots(2)$$

(2) $\Rightarrow$ $a_i dy - b_i dx = 0$ $\Rightarrow$ $a_i y - b_i x = \lambda$

Also, (2) $\Rightarrow$ $\dfrac{dz}{z} = -\dfrac{c_i}{a_i} dx$ $\Rightarrow$ $\log z = -\dfrac{c_i}{a_i} x + \log \mu$

$\Rightarrow$ $z = \mu e^{-c_i x/a_i}$ $\Rightarrow$ $z = e^{-c_i x/a_i} \phi_i(a_i y - b_i x)$ (Putting $\mu = \phi_i(\lambda)$)

This is true only when $a_i \neq 0$.

If $b_i \neq 0$, then by taking IInd and IIIrd fractions of (2), we can show that

$z = e^{-c_i y/b_i} \phi_i(a_i y - b_i x)$ is a solution of (1).

$\therefore$ $z = e^{-c_1 x/a_1} \phi_1(a_1 y - b_1 x)$, $z = e^{-c_2 x/a_2} \phi_2(a_2 y - b_2 x)$, ...,

$z = e^{-c_n x/a_n} \phi_n(a_n y - b_n x)$ are solutions of (1).

$$\therefore \quad \mathbf{z = e^{-c_1 x/a_1} \phi_1(a_1 y - b_1 x) + e^{-c_2 x/a_2} \phi_2(a_2 y - b_2 x) + ... + e^{-c_n x/a_n} \phi_n(a_n y - b_n x)}$$

is also a solution of (1). Since this solution contains n arbitrary functions ϕ_1, ϕ_2, ..., ϕ_n, this solution represents the general solution of the given equation. Here we have assumed that a_1, a_2, ..., a_n are all non-zero constants.

Case II. Factors are not distinct

Let the first two factors be same and all others distinct. In this case, the solution of (1) can be written as $z = e^{-c_1 x/a_1} (\phi_1 + \phi_2)(a_1 y - b_1 x) + ... + e^{-c_n x/a_n} \phi_n(a_n y - b_n x)$.

This solution contains only $n - 1$ arbitrary functions ϕ_1, ϕ_2, ϕ_3, ..., ϕ_n.

$\therefore$ This is not a general solution.

(1) $\Rightarrow$ $(a_1 D + b_1 D' + c_1)^2 (a_3 D + b_3 D' + c_3)$ ...... $(a_n D + b_n D' + c_n) z = 0$ ...(3)

Equation (3) shows that for $3 \leq i \leq n$, the solution of $(a_i D + b_i D' + c_i) z = 0$ is also a solution of (3) and hence of (1).

$\therefore$ $z = e^{-c_i x/a_i} \phi_i(a_i y - b_i x)$, $3 \leq i \leq n$ is a solution of (1), provided $a_i \neq 0$, $3 \leq i \leq n$.

$$\therefore \quad z = e^{-c_3 x/a_3} \phi_3(a_3 y - b_3 x) + ... + e^{-c_n x/a_n} \phi_n(a_n y - b_n x) \quad ...(4)$$

is a solution of (1).

The solution of $(a_1 D + b_1 D' + c_1)^2 z = 0$ is also a solution of (3) and hence of (1).

$$(a_1 D + b_1 D' + c_1)^2 z = 0 \Rightarrow (a_1 D + b_1 D' + c_1)(a_1 D + b_1 D' + c_1) z = 0 \quad ...(5)$$

Let $(a_1 D + b_1 D' + c_1) z = u$ ...(6)

$\therefore$ (5) $\Rightarrow$ $(a_1 D + b_1 D' + c_1) u = 0$

$\Rightarrow$ $u = e^{-c_1 x/a_1} \psi_1(a_1 y - b_1 x)$

where ψ_1 is arbitrary. We are assuming that $a_1 \neq 0$.

$\therefore$ (6) $\Rightarrow$ $a_1 p + b_1 q + c_1 z = e^{-c_1 x/a_1} \psi_1(a_1 y - b_1 x)$

$\Rightarrow$ $a_1 p + b_1 q = e^{-c_1 x/a_1} \psi_1(a_1 y - b_1 x) - c_1 z$

$$\Rightarrow \quad \frac{dx}{a_1} = \frac{dy}{b_1} = \frac{dz}{e^{-c_1 x/a_1} \psi_1(a_1 y - b_1 x) - c_1 z} \quad ...(7)$$

(7) $\Rightarrow$ $a_1 dy - b_1 dx = 0$ $\Rightarrow$ $a_1 y - b_1 x = c$

NOTES

Taking the first and third fractions of (7), we get

$$\frac{dz}{dx} = \frac{1}{a_1}(e^{-c_1x/a_1}\ \psi_1(a_1y - b_1x) - c_1z)$$

$$\Rightarrow \qquad \frac{dz}{dx} + \frac{c_1}{a_1}z = \frac{1}{a_1}e^{-c_1x/a_1}\ \psi_1(a_1y - b_1x) \qquad ...(8)$$

This is a linear equation.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{c_1}{a_1}dx} = e^{c_1x/a_1}$$

$\therefore$ Solution of (8) is

$$z \cdot e^{c_1x/a_1} = \int \frac{1}{a_1}e^{-c_1x/a_1}\ \psi_1(a_1y - b_1x) \,.\, e^{c_1x/a_1}\,dx$$

$$= \frac{1}{a_1}\int \psi_1(c)\,dx = \frac{x\psi_1(c)}{a_1} + d$$

$$\Rightarrow \qquad ze^{c_1x/a_1} = \frac{x}{a_1}\psi_1(a_1y - b_1x) + d$$

$$\Rightarrow \qquad ze^{c_1x/a_1} = x\,\phi_1(a_1y - b_1x) + \phi_2(a_1y - b_1x)$$

$$\left(\text{Taking } \phi_1 = \frac{1}{a_1}\psi_1 \text{ and } d = \phi_2(c)\right)$$

$$\therefore \qquad z = e^{-c_1x/a_1}\ (x\,\phi_1(a_1y - b_1x) + \phi_2(a_1y - b_1x)) \qquad ...(9)$$

Combining (4) and (9),

$$\mathbf{z = e^{-c_1x/a_1}\ (x\,\phi_1(a_1y - b_1x) + \phi_2(a_1y - b_1x))}$$

$$\mathbf{+\ e^{-c_3x/a_3}\ \phi_3(a_3y - b_3x) + ... + e^{-c_nx/a_n}\ \phi_n(a_ny - b_nx).}$$

Since this solution contains n arbitrary functions ϕ_1, ϕ_2, ..., ϕ_n, this solution represents the general solution of the given equation.

Remark. If the factor $a_1x + b_1y + c_1$ is repeated r times, then the corresponding part of the general solution is e^{-c_1x/a_1} $(\phi_1(a_1y - b_1x) + x\,\phi_2(a_1y - b_1x) + ... + x^{r-1}\,\phi_r(a_1y - b_1x))$, where ϕ_1, ϕ_2, ..., ϕ_r are arbitrary functions.

WORKING STEPS FOR SOLVING PROBLEMS

Step I. *Express f(D, D′) as the product of linear factors in D and D′.*

Step II. *Corresponding to each non-repeated factor aD + bD′ + c, the part of G.S. is* $e^{-cx/a}\ \phi(ay - bx)$, *if* $a \neq 0$ *or* $e^{-cy/b}\ \phi(ay - bx)$ *if* $b \neq 0$.

Step III. *Corresponding to each repeated factor aD + bD′ + c, r times, the part of G.S. is*

$$e^{-cx/a}\sum_{i=1}^{r} x^{i-1}\ \phi_i(ay - bx),\ \text{if } a \neq 0 \text{ or } e^{-cy/b}\sum_{i=1}^{r} y^{i-1}\ y_i(ay - bx),\ \text{if } b \neq 0.$$

SOLVED EXAMPLES

Example 1. *Find the general solution of following partial differential equations:*

(*i*) $t + s + q = 0$ (*ii*) $DD'(D - 2D' - 3)\, z = 0$

(*iii*) $(DD' + aD + bD' + ab)z = 0$ (*iv*) $r + 2s + t + 2p + 2q + z = 0.$

Sol. (*i*) We have $t + s + q = 0$.

$\Rightarrow$ $(D'^2 + DD' + D')\, z = 0 \quad \Rightarrow \quad (0.D + 1.D' + 0)(D' + D + 1)\, z = 0$ ...(1)

$\therefore$ The general solution of (1) is

$$z = e^{-0y/1}\, \phi_1(0y - 1.x) + e^{-1.x/1}\, \phi_2(1.y - 1.x)$$

or $$\mathbf{z = \phi_1(-x) + e^{-x}\, \phi_2(y - x),}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Remark. $\phi_1(-x)$ is also a function of x and can also be written as $\psi(x)$ for some arbitrary function ψ.

(*ii*) We have $DD'(D - 2D' - 3)\, z = 0.$ ...(1)

$\Rightarrow$ $(1.D + 0.D' + 0)(0.D + 1.D' + 0)(D - 2D' - 3)z = 0$

$\therefore$ The general solution of (1) is

$$z = e^{-0.x/1}\, \phi_1(1.y - 0.x) + e^{-0.y/1}\, \phi_2(0.y - 1.x) + e^{3x/1}\, \phi_3(1.y + 2.x)$$

or $$\mathbf{z = \phi_1(y) + \phi_2(-x) + e^{3x}\, \phi_3(2x + y)},$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

(*iii*) We have $(DD' + aD + bD' + ab)\, z = 0.$

$\Rightarrow$ $(D + b)(D' + a)\, z = 0 \quad \Rightarrow \quad (1.D + 0.D' + b)(0.D + 1.D' + a)z = 0$ ...(1)

$\therefore$ The general solution of (1) is

$$z = e^{-bx/1}\, \phi_1(1.y - 0.x) + e^{-ay/1}\, \phi_2(0.y - 1.x)$$

or $$\mathbf{z = e^{-bx}\, \phi_1(y) + e^{-ay}\, \phi_2(-x),}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

(*iv*) We have $r + 2s + t + 2p + 2q + z = 0$ ...(1)

$\Rightarrow$ $(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)\, z = 0$

$\Rightarrow$ $[(D + D')^2 + 2(D + D') + 1]\, z = 0$

$\Rightarrow$ $(D + D' + 1)^2\, z = 0$

$\therefore$ The general solution of (1) is

$$z = e^{-1.x/1}\, (\phi_1(1.y - 1.x) + x\, \phi_2(1.y - 1.x))$$

or $$\mathbf{z = e^{-x}\, (\phi_1(y - x) + x\, \phi_2(y - x)),}$$

where ϕ_1, ϕ_2 are arbitrary functions.

GENERAL SOLUTION OF IRREDUCIBLE NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION f(D, D′)z = 0 WITH CONSTANT COEFFICIENTS

Let $f(D, D')\, z = 0$ ...(1)

be an irreducible non-homogeneous linear partial differential equation with constant coefficients.

NOTES

Let $z = ce^{ax+by}$.

$\therefore$ $D^r D^s z = D^r D^s(ce^{ax+by}) = ca^r b^s e^{ax+by} = a^r b^r ce^{ax+by}$

$\therefore$ $f(D, D') z = f(a, b) ce^{ax+by}$

$\therefore$ $z = ce^{ax+by}$ is a solution of (1) if $f(a, b) = 0$ and the constant c is arbitrary. Let (a_i, b_i) be one of infinitely many pairs satisfying $f(a_i, b_i) = 0$.

$\therefore$ $z = c_i e^{a_i x + b_i y}$ is a solution of (1) for each i.

$\therefore$ The general solution of (1) is $\mathbf{z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}}$, **where** $\mathbf{f(a_i, b_i) = 0}$.

Remark. For the above irreducible partial differential equation (1), the general solution has been written in terms of arbitrary constants.

WORKING STEPS FOR SOLVING PROBLEMS

Step I. *Express the given equation in the form f(D, D′) g(D, D′) z = 0, where f(D, D′) is expressible as a product of linear factors in D and D′ and g(D, D′) is irreducible.*

Step II. *Write the solution of f(D, D′)z = 0 in terms of arbitrary functions.*

Step III. *Write the solution of g(D, D′)z = 0 in terms of arbitrary constants.*

Step IV. *Add both general solutions to get the general solution of the given equation.*

Example 2. *Find the general solution of the following partial differential equations :*

(*i*) $(D^2 + D + D') z = 0$ (*ii*) $(2D^4 - 3D^2D' + D'^2) z = 0$

(*iii*) $(D + 2D' - 3)(D^2 + D')z = 0$ (*iv*) $(D' + 3D)^2 (D^2 + 5D + D') z = 0$.

Sol. (*i*) We have $(D^2 + D + D') z = 0$. ...(1)

$D^2 + D + D'$ is irreducible

Let $z = ce^{ax+by}$ be a solution of (1).

$\therefore$ $(a^2 + a + b) ce^{ax+by} = 0$

$\Rightarrow$ $a^2 + a + b = 0 \Rightarrow b = -(a^2 + a)$

$\therefore$ $z = ce^{ax-(a^2+a)y}$, where a and c are arbitrary constants.

$\therefore$ The general solution of (1) is $z = \sum_{i=1}^{\infty} \mathbf{c_i e^{a_i x - (a_i^2 + a_i)y}}$, where a_i and c_i are arbitrary constants.

(*ii*) We have $(2D^4 - 3D^2D' + D'^2) z = 0$. ...(1)

$\Rightarrow$ $(2D^2 - D')(D^2 - D') z = 0$

The factor $2D^2 - D'$ is irreducible.

Let $z = ce^{ax+by}$ be a solution of $(2D^2 - D') z = 0$.

$\therefore$ $(2a^2 - b) ce^{ax+by} = 0 \Rightarrow 2a^2 - b = 0$ or $b = 2a^2$

$\therefore$ $z = ce^{ax + 2a^2 y}$

$\therefore$ Corresponding to $2D^2 - D'$, the part of general solution of (1) is $\sum_{i=1}^{\infty} c_i e^{a_i x + 2a_i^2 y}$.

The factor $D^2 - D'$ is also irreducible.

Let $z = c' e^{a'x + b'y}$ be a solution of $(D^2 - D')\, z = 0$.

$\therefore$ $\quad (a'^2 - b')\, c' e^{a'x+b'y} = 0$

$\Rightarrow$ $\quad a'^2 - b' = 0$ or $b' = a'^2$

$\therefore$ $\quad z = c' e^{a'x + a'^2 y}$

$\therefore$ Corresponding to $D^2 - D'$, the part of general solution of (1) is $\sum_{i=1}^{\infty} c_i' e^{a'_i x + a'^2_i y}$

$\therefore$ The general solution of (1) is $\mathbf{z = \sum_{i=1}^{\infty} c_i e^{a_i x + 2a_i^2 y} + \sum_{i=1}^{\infty} c_i' e^{a_i' x + a_i'^2 y}}$, where a_i, c_i, a_i', c_i' are arbitrary constants.

(*iii*) We have $\quad (D + 2D' - 3)(D^2 + D')\, z = 0.$ ...(1)

The factor $D + 2D' - 3$ is linear.

$\therefore$ Corresponding to $D + 2D' - 3$, the part of general solution of (1) is $e^{-(-3)x/1} \phi_1(1.y - 2.x)$ *i.e.* $e^{3x}\phi_1(y - 2x)$.

The factor $D^2 + D'$ is irreducible.

Let $z = ce^{ax+by}$ be a solution of $(D^2 + D')\, z = 0$.

$\therefore$ $\quad (a^2 + b)\, ce^{ax+by} = 0.$

$\Rightarrow$ $\quad a^2 + b = 0$ or $b = -a^2$

$\therefore$ $\quad z = ce^{ax - a^2 y}$

$\therefore$ Corresponding to $D^2 + D'$, the part of general solution of (1) is $\sum_{i=1}^{\infty} c_i e^{a_i x - a_i^2 y}$.

$\therefore$ The general solution of (1) is $\mathbf{z = e^{3x}\, \phi_1(y - 2x) + \sum_{i=1}^{\infty} c_i e^{a_i x - a_i^2 y}}$, where ϕ_1 is arbitrary function and a_i, c_i are arbitrary constants.

(*iv*) We have $\quad (D' + 3D)^2\, (D^2 + 5D + D')\, z = 0.$ ...(1)

The factor $(3D + D')^2$ is linear repeated.

$\therefore$ Corresponding to $(3D + D')^2$, the part of general solution of (1) is

$$e^{-0.x/3}\, (\phi_1(3.y - 1.x) + x\phi_2(3.y - 1.x)), \quad i.e., \quad \phi_1(3y - x) + x\phi_2(3y - x).$$

The factor $D^2 + 5D + D'$ is irreducible.

Let $z = ce^{ax+by}$ be a solution of $(D^2 + 5D + D')\, z = 0$.

$\therefore$ $\quad (a^2 + 5a + b)\, ce^{ax+by} = 0$

$\Rightarrow$ $\quad a^2 + 5a + b = 0$ or $b = -(a^2 + 5a)$

$\therefore$ $\quad z = ce^{ax - (a^2 + 5a)y}$

$\therefore$ Corresponding to $D^2 + 5D + D'$, the part of general solution of (1) is $\sum_{i=1}^{\infty} c_i e^{a_i x - (a_i^2 + 5a_i)y}$. 250

NOTES

$\therefore$ The general solution of (1) is $\mathbf{z = \phi_1(3y - x) + x\,\phi_2(3y - x) + \sum_{i=1}^{\infty} c_i e^{a_i x - (a_i^2 + 5a_i)y}}$, where ϕ_1, ϕ_2 are arbitrary functions and a_i, c_i are arbitrary constants.

EXERCISE A

Find the general solution of the following partial differential equations :

1. $(D + D' - 1)(D + 2D' - 2)z = 0$
2. $(D + 1)(D + D' - 1)z = 0$
3. $(D^2 - DD' - 2D)z = 0$
4. $(D^2 - D'^2 + D - D')z = 0$
5. $(D - D' - 1)(D - D' - 2)\ z = 0$
6. $(D^2 - DD' + D' - 1)z = 0$
7. $s + p - q - z = 0$
8. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$
9. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = 0$
10. $(D - 3D' - 2)^2\ z = 0$
11. $(D - D'^2)\ z = 0$
12. $(2D^2 - D'^2 + D)\ z = 0$
13. $(D^2 + DD' + D + D' + 1)\ z = 0$
14. $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$
15. $(2D - 3D' + 5)^2\ (D^2 + 3D')\ z = 0.$

Answers

1. $z = e^x\phi_1(y - x) + e^{2x}\phi_2(y - 2x)$
   or $z = e^y\phi_1(y - x) + e^{2y}\phi_2(y - 2x)$
2. $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$
3. $z = \phi_1(y) + e^{2x}\phi_2(y + x)$
4. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$
5. $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$
6. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)$
7. $z = e^x\phi_1(y) + e^{-y}\phi_2(-x)$
8. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$
9. $z = \phi_1(y + 2x) + \phi_2(y - 3x)$
10. $z = e^{2x}\ (\phi_1(y + 3x) + x\phi_2(y + 3x))$
11. $z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y}$
12. $\sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$, where $2a_i^2 - b_i^2 + a_i = 0$
13. $z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$, where $a_i^2 + a_i b_i + a_i + b_i + 1 = 0$
14. $z = e^x\ \phi(y + 2x) + \sum_{i=1}^{\infty} c_i e^{(1 + 2b_i^2)x + b_i y}$
15. $z = e^{-5x/2}\ (\phi_1(2y + 3x) + x\phi_2(2y + 3x)) + \sum_{i=1}^{\infty} c_i e^{a_i x - a_i^2 y/3}$.

GENERAL SOLUTION OF NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let $$f(D, D')\ z = F(x, y) \qquad ...(1)$$
be a non-homogeneous linear partial differential equation with constant coefficients.

Let u be the general solution of $f(D, D')\, z = 0$.

$\therefore \quad f(D, D')\, u = 0 \quad ...(2)$

Let v be a particular integral of $f(D, D')\, z = F(x, y)$.

$\therefore \quad f(D, D')\, v = F(x, y) \quad ...(3)$

Now $f(D, D')\,(u + v) = f(D, D')\, u + f(D, D')\, v = 0 + F(x, y) = F(x, y)$.

$\therefore$ $u + v$ is a solution of $f(D, D')\, z = F(x, y)$. Since u is the general solution of the equation $f(D, D')\, z = 0$, the solution $u + v$ of the equation $f(D, D')\, z = F(x, y)$ is the general solution of the equation $f(D, D')\, z = F(x, y)$.

The general solution u of the equation $f(D, D')\, z = 0$ is called the **complementary function (C.F.)** of the equation $f(D, D')\, z = F(x, y)$.

$\therefore$ **The general solution of the equation f(D, D′) z = F(x, y) is obtained by adding the general solution of the equation f(D, D′)z = 0 to any particular integral of the equation f(D, D′)z = F(x, y).**

PARTICULAR INTEGRAL OF f(D, D′)z = F(x, y)

Let $\quad f(D, D')\, z = F(x, y) \quad ...(1)$

be a non-homogeneous linear partial differential equation with constant coefficients.

Since, $f(D, D')\left[\dfrac{1}{f(D, D')}F(x, y)\right] = F(x, y)$, the function $\dfrac{1}{f(D, D')}F(x, y)$ is a particular integral of the equation $f(D, D')\, z = F(x, y)$.

PARTICULAR INTEGRAL WHEN F(x, y) IS SUM OR DIFFERENCE OF TERMS OF THE FORM $x^m y^n$

If $F(x, y)$ is sum or difference of the terms of the form $x^m y^n$, then the particular integral $\dfrac{1}{f(D, D')}F(x, y)$ of the differential equation $f(D, D')\, z = F(x, y)$ is obtained by expanding $\dfrac{1}{f(D, D')}$ in an infinite series in ascending powers of either D or D′. The particular integrals obtained in the above mentioned two method may not be identical. Any one of the two particular integrals may be used.

SOLVED EXAMPLES

Example 3. *Find the general solution of the following partial differential equations :*

(*i*) $(D^2 - D' - 1)z = x^2 y$ $\qquad$ (*ii*) $(D^2 - D'^2 - 3D + 3D')z = xy$.

Sol. (*i*) We have $\quad (D^2 - D' - 1)\, z = x^2 y. \quad ...(1)$

$D^2 - D' - 1$ is irreducible.

$\therefore \quad \text{C.F.} = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$, where $a_i^2 - b_i - 1 = 0$ or $b_i = a_i^2 - 1$

NOTES

$\therefore$ $$\text{C.F.} = \sum_{i=1}^{\infty} c_i e^{a_i x + (a_i^2 - 1)y}.$$

$$\text{P.I.} = \frac{1}{D^2 - D' - 1} x^2 y = -\frac{1}{1 + (D' - D^2)} x^2 y$$
$$= -(1 + (D' - D^2))^{-1} x^2 y$$
$$= -(1 - (D' - D^2) + (D' - D^2)^2 - \ldots\ldots) x^2 y$$
$$= -(1 - D' + D^2 - 2D'D^2 + \ldots\ldots) x^2 y$$
$$= -(x^2 y - x^2 + 2y - 4 + 0 + \ldots\ldots) = -x^2 y + x^2 - 2y + 8.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = \sum_{i=1}^{\infty} c_i e^{a_i x + (a_i^2 - 1)y} - x^2 y + x^2 - 2y + 8},$$ where a_i and c_i are arbitrary constants.

(*ii*) We have $(D^2 - D'^2 - 3D + 3D')\, z = xy$. ...(1)

$D^2 - D'^2 - 3D + 3D' = (D - D')(D + D' - 3) = (1.D - 1.D' + 0)(1.D + 1.D' - 3)$

$\therefore$ C.F. $= e^{-0.x/1}\, \phi_1(1.y + 1.x) + e^{-(-3)x/1}\, \phi_2(1.y - 1.x) = \phi_1(y + x) + e^{3x}\phi_2(y - x)$.

$$\text{P.I.} = \frac{1}{(D - D')(D + D' - 3)} xy$$
$$= \frac{1}{D}\left(1 - \frac{D'}{D}\right)^{-1} \cdot \frac{1}{-3}\left(1 - \left(\frac{D}{3} + \frac{D'}{3}\right)\right)^{-1} xy$$
$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\ldots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \left(\frac{D}{3} + \frac{D'}{3}\right)^2 + \ldots\ldots\right) xy$$
$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\ldots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\ldots\right) xy$$
$$= -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \frac{D'}{D} + \frac{D'}{3} + \ldots\ldots\right) xy$$
$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{x}{3} + \frac{2}{9} + \frac{x^2}{2} + \frac{x}{3} + 0 + \ldots\ldots\right)$$
$$= -\frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{6} + \frac{2x}{9} + \frac{x^3}{6} + \frac{x^2}{6}\right)$$
$$= -\frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x\right).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - \frac{1}{3}\left(\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x\right)},$$

where ϕ_1, ϕ_2 are arbitrary functions.

Alternative method of finding P.I.

$$\text{P.I.} = \frac{1}{(D - D')(D + D' - 3)} xy$$

$$= -\frac{1}{3(D-D')}\left(1-\frac{D+D'}{3}\right)^{-1} xy$$

$$= -\frac{1}{3(D-D')}\left(1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\ldots\right) xy$$

$$= -\frac{1}{3(D-D')}\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{2DD'}{9}+\ldots\right) xy$$

$$= -\frac{1}{3D\left(1-\frac{D'}{D}\right)}\left(xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}\right)$$

$$= -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}\right)$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}+\ldots\right)\left(xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}\right)$$

$$= -\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}+\frac{1}{D}\left(x+\frac{1}{3}\right)+0+\ldots\right)$$

$$= -\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}\right)-\frac{1}{3D^2}\left(x+\frac{1}{3}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{6}+\frac{2x}{9}\right)-\frac{1}{3}\left(\frac{x^3}{6}+\frac{x^2}{6}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2}{9}x\right).$$

Remark. In solving problems, the second method is found comparatively easier and straight forward.

EXERCISE B

Find the general solution of the following partial differential equations:

1. $r - s + p = 1$
2. $(D - D' - 1)(D - D' - 2)\, z = x$
3. $s + p - q = z + xy$
4. $(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = xy$
5. $(D^2 - D'^2 + D + 3D' - 2)\, z = x^2y$
6. $D(D + D' - 1)(D + 3D' - 2)\, z = x^2 - 4xy + 2y^2$.

Answers

1. $z = \phi_1(y) + e^{-x}\,\phi_2(y + x) + x$
2. $z = e^{x}\,\phi_1(y + x) + e^{2x}\,\phi_2(y + x) + \frac{x}{2} + \frac{3}{4}$
3. $z = e^{x}\,\phi_1(y) + e^{-y}\,\phi_2(x) - xy - y + x + 1$
4. $z = \phi_1(y - x) + e^{-2x}\,\phi_2(y + 2x) + \frac{x^2y}{4} - \frac{xy}{4} - \frac{x^3}{12} + \frac{3x^2}{8} - \frac{x}{2}$
5. $z = e^{-2x}\,\phi_1(y + x) + e^{x}\,\phi_2(y - x) - \frac{1}{2}\left(x^2y + xy + \frac{3}{2}x^2 + \frac{3}{2}y + 3x + \frac{21}{4}\right)$
6. $z = \phi_1(y) + e^{x}\,\phi_2(y - x) + e^{2x}\,\phi_3(y - 3x) + \frac{1}{12}(2x^3 - 12x^2y + 12xy^2 - 21x^2 + 24xy + 3x)$.

PARTICULAR INTEGRAL WHEN F(x, y) IS OF THE FORM e^{ax+by}

NOTES

Theorem. **If f(D, D′) be a function of D and D′, then** $\dfrac{1}{f(D, D')} e^{ax+by} = \dfrac{1}{f(a, b)} e^{ax+by}$ **provided** $f(a, b) \neq 0$.

Proof. We have $D^r e^{ax+by} = a^r e^{ax+by}$

$$D'^s e^{ax+by} = b^s e^{ax+by}$$

and $$D^r D'^s e^{ax+by} = a^r b^s e^{ax+by}.$$

$$\therefore \quad f(D, D')\, e^{ax+by} = f(a, b)\, e^{ax+by}$$

Dividing both sides by $f(a, b)$, we get $f(D, D') \dfrac{e^{ax+by}}{f(a,b)} = e^{ax+by}$

$\therefore$ By definition of the inverse operator $\dfrac{1}{f(D, D')}$, we have $\dfrac{1}{f(D, D')} e^{ax+by} = \dfrac{e^{ax+by}}{f(a, b)}$.

Remark 1. The above result is not true for any general function $\phi(ax + by)$ of $ax + by$.

Remark 2. The method $\dfrac{1}{f(D, D')} e^{ax+by} = \dfrac{1}{f(a, b)} e^{ax+by}$ is applicable only when $f(a, b) \neq 0$. The case when $f(a, b) = 0$ will be discussed a little later.

SOLVED EXAMPLES

Example 4. *Find the general solution of the following partial differential equations :*

(*i*) $(DD' + aD + bD' + ab)z = e^{mx+ny}$ (*ii*) $(D^2 + D' + 4)z = e^{4x-y}$.

Sol. (*i*) We have $(DD' + aD + bD' + ab)z = e^{mx+ny}$. ...(1)

$$DD' + aD + bD' + ab = (D + b)(D' + a) = (1.D + 0.D' + b)(0.D + 1.D' + a)$$

$\therefore$ C.F. $= e^{-bx/1} \phi_1(1.y - 0.x) + e^{-ay/1} \phi_2(0.y - 1.x) = e^{-bx} \phi_1(y) + e^{-ay} \phi_2(-x)$.

$$\text{P.I.} = \frac{1}{DD' + aD + bD' + ab} e^{mx+ny} \quad ...(2)$$

$D = m, D' = n \Rightarrow DD' + aD + bD' + ab = mn + am + bn + ab$

$= (m + b)(n + a) \neq 0$ (Assuming $m \neq -b, n \neq -a$)

$\therefore$ (2) $\Rightarrow$ $$\text{P.I.} = \frac{1}{(m+b)(n+a)} e^{mx+ny}$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = e^{-bx} \phi_1(y) + e^{-ay} \phi_2(-x) + \frac{e^{mx+ny}}{(m+b)(n+a)}},$$

where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) We have $(D^2 + D' + 4)\, z = e^{4x-y}$. ...(1)

$D^2 + D' + 4$ is irreducible.

$$\therefore \qquad \text{C.F.} = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}, \text{ where } a_i^2 + b_i + 4 = 0$$

$$= \sum_{i=1}^{\infty} c_i e^{a_i x - (a_i^2 + 4)y}.$$

$$\text{P.I.} = \frac{1}{D^2 + D' + 4} e^{4x - y} \qquad \ldots(2)$$

$$D = 4, D' = -1 \quad \Rightarrow \quad D^2 + D' + 4 = (4)^2 + (-1) + 4 = 19 \neq 0$$

$$\therefore \quad (2) \Rightarrow \text{P.I.} = \frac{1}{19} e^{4x - y}$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is $\mathbf{z = \sum_{i=1}^{\infty} c_i{}^{a_i x - (a_i^2 + 4)y} + \frac{1}{19} e^{4x - y}}$, where a_i and c_i are arbitrary constants.

EXERCISE C

Find the general solution of the following partial differential equations :

1. $(D^2 - DD' - 2D)\, z = e^{2x + y}$
2. $(D^2 - D'^2 + D - D')z = e^{2x + 3y}$
3. $(D^2 - 4DD' + D - 1)\, z = e^{3x - 2y}$
4. $(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = e^{2x + 3y}$
5. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x - y}$
6. $(D^3 - 3DD' + D + 1)\, z = e^{2x + 3y}$.

Answers

1. $z = \phi_1(y) + e^{2x}\, \phi_2(y + x) - \frac{1}{2} e^{2x + y}$
2. $z = \phi_1(y + x) + e^{-x}\, \phi_2(y - x) - \frac{1}{6} e^{2x + 3y}$
3. $z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y} + \frac{1}{35} e^{3x - 2y}$, where $a_i^2 - 4a_i b_i + a_i - 1 = 0$
4. $z = \phi_1(y - x) + e^{-2x}\, \phi_2(y + 2x) - \frac{1}{10} e^{2x + 3y}$
5. $z = e^{-2x}\, \phi_1(y + x) + e^{x}\, \phi_2(y - x) - \frac{1}{4} e^{x - y}$
6. $z = \sum_{i=1}^{\infty} c e^{a_i x + b_i y} - \frac{1}{7} e^{2x + 3y}$, where $a_i^3 - 3a_i b_i + a_i + 1 = 0$.

PARTICULAR INTEGRAL WHEN F(x, y) IS OF THE FORM sin (ax + by) OR cos (ax + by)

Let the given partial differential equation be

$$f(D, D')\, z = F(x, y), \text{ where } F(x, y) = \sin(ax + by) \text{ (or } \cos(ax + by)).$$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} \sin(ax + by) \text{ (or } \cos(ax + by))$$

This is evaluated by putting $D^2 = -a^2$, $DD' = -ab$ and $D'^2 = -b^2$, provided the denominator is not zero.

NOTES

SOLVED EXAMPLES

Example 5. *Find the general solution of the following partial differential equations :*

(*i*) $(D + D')(D + D' - 2)z = \sin(x + 2y)$

(*ii*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y)$.

Sol. (*i*) We have $(D + D')(D + D' - 2)\, z = \sin(x + 2y)$. ...(1)

$\Rightarrow \quad (1 \,.\, D + 1 \,.\, D' + 0)(1 \,.\, D + 1 \,.\, D' - 2)z = \sin(x + 2y)$

$\therefore \quad \text{C.F.} = e^{-0.x/1}\, \phi_1(1.y - 1.x) + e^{-(-2)x/1}\, \phi_2(1.y - 1.x)$

$\therefore \quad \text{C.F.} = \phi_1(y - x) + e^{2x}\, \phi_2(y - x)$.

$$\text{P.I.} = \frac{1}{(D + D')(D + D' - 2)} \sin(x + 2y) \qquad ...(2)$$

Here $a = 1,\ b = 2$

$\therefore \quad D^2 = -a^2 = -(1)^2 = -1, \quad DD' = -ab = -(1)(2) = -2,$

$D'^2 = -b^2 = -(2)^2 = -4$.

$$\therefore \quad (2) \Rightarrow \text{P.I.} = \frac{1}{(D + D')^2 - 2(D + D')} \sin(x + 2y)$$

$$= \frac{1}{D^2 + D'^2 + 2DD' - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{1}{-1 - 4 + 2(-2) - 2D - 2D'} \sin(x + 2y)$$

$$= -\frac{1}{2D + 2D' + 9} \sin(x + 2y)$$

$$= -\frac{2D + 2D' - 9}{(2D + 2D')^2 - 81} \sin(x + 2y)$$

$$= -\frac{2D + 2D' - 9}{4D^2 + 8DD' + 4D'^2 - 81} \sin(x + 2y)$$

$$= -\frac{2D + 2D' - 9}{4(-1) - 8(1)(2) - 4(-4) - 81} \sin(x + 2y)$$

$$= \frac{2D + 2D' - 9}{117} \sin(x + 2y)$$

$$= -\frac{1}{117}[2\cos(x + 2y) + 4\cos(x + 2y) - 9\sin(x + 2y)]$$

$$= -\frac{1}{117}[6\cos(x + 2y) - 9\sin(x + 2y)]$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = \phi_1(y - x) + e^{2x}\, \phi_2(y - x) - \frac{1}{117}[6\cos(x + 2y) - 9\sin(x + 2y)],}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) We have $(D^2 - DD' - 2D'^2 + 2D + 2D')\, z = e^{2x+3y} + \sin(2x + y)$. ...(1)

$D^2 - DD' - 2D'^2 + 2D + 2D' = (D + D')(D - 2D') + 2(D + D')$

$= (D + D')(D - 2D' + 2) = (1.D + 1.D' + 0)(1.D - 2D' + 2)$

$\therefore \quad \text{C.F.} = e^{-0.x/1}\, \phi_1(1.y - 1.x) + e^{-2x/1}\, \phi_2(1.y - (-2)x)$

$\therefore$ $$\text{C.F.} = \phi_1(y - x) + e^{-2x}\,\phi_2(y + 2x).$$

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\,(e^{2x+3y} + \sin(2x + y))$$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\,e^{2x+3y} + \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x + y)$$

$$= \frac{1}{(2)^2 - 2(3) - 2(3)^2 + 2(2) + 2(3)}\,e^{2x+3y} + \frac{1}{(-4) + 2(1) - 2(-1) + 2D + 2D'}\sin(2x + y)$$

$$= -\frac{1}{10}\,e^{2x+3y} + \frac{1}{2D + 2D'}\sin(2x + y)$$

$$= -\frac{1}{10}\,e^{2x+3y} + \frac{D - D'}{2(D^2 - D'^2)}\sin(2x + y)$$

$$= -\frac{1}{10}\,e^{2x+3y} + \frac{D - D'}{2(-4 + 1)}\sin(2x + y)$$

$$= -\frac{1}{10}\,e^{2x+3y} - \frac{1}{6}(2\cos(2x + y) - \cos(2x + y))$$

$$= -\frac{1}{10}\,e^{2x+3y} - \frac{1}{6}\cos(2x + y).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = \phi_1(y - x) + e^{-2x}\,\phi_2(y + 2x) - \frac{1}{10}\,e^{2x+3y} - \frac{1}{6}\cos(2x + y),}$$

where ϕ_1, ϕ_2 are arbitrary functions.

EXERCISE D

Find the general solution of the following partial differential equations :

1. $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$
2. $(D - D' - 1)(D - D' - 2)\,z = \sin(2x + 3y)$
3. $(D^2 - DD' - 2D)\,z = \sin(3x + 4y)$
4. $(2DD' + D'^2 - 3D')\,z = 3\cos(3x - 2y)$
5. $(D - D'^2)z = \cos(x - 3y)$
6. $(D^2 + D')(D - D' - D'^2)\,z = \sin(2x + y)$.

Answers

1. $z = e^x\,\phi_1(y) + e^{-x}\,\phi_2(y + x) + \frac{1}{2}\sin(x + 2y)$
2. $z = e^x\,\phi_1(y + x) + e^{2x}\,\phi_2(y + x) + \frac{1}{10}[\sin(2x + 3y) - 3\cos(2x + 3y)]$
3. $z = \phi_1(y) + e^{2x}\,\phi_2(y + x) + \frac{1}{15}[\sin(3x + 4y) + 2\cos(3x + 4y)]$
4. $z = \phi_1(x) + e^{3x/2}\,\phi_2(2y - x) + \frac{3}{50}[4\cos(3x - 2y) + 3\sin(3x - 2y)]$
5. $z = \sum_{i=1}^{\infty} c_i e^{b_i^2 x + b_i y} + \frac{1}{82}[\sin(x - 3y) + 9\cos(x - 3y)]$
6. $z = \sum_{i=1}^{\infty} c_i e^{a_i x - a_i^2 y} + \sum_{i=1}^{\infty} k_i e^{(b_i + b_i^2)x + b_i y} - \frac{1}{34}[5\sin(2x + y) - 3\cos(2x + y)]$.

NOTES

PARTICULAR INTEGRAL WHEN F(x, y) IS OF THE FORM e^{ax+by} V(x, y)

Let the given partial differential equation be

$$f(D, D')\, z = F(x, y), \text{ where } F(x, y) = e^{ax+by}\, V(x, y).$$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')}\, e^{ax+by}\, V(x, y)$$

This is evaluated by using the formula :

$$\frac{1}{f(D, D')}\, e^{ax+by}\, V(x, y) = e^{ax+by}\, \frac{1}{f(D+a, D'+b)}\, V(x, y).$$

Remark. If $F(x, y) = e^{ax+by}$ and $f(a, b) = 0$, then we cannot write $\frac{1}{f(D, D')} e^{ax+by}$ as $\frac{1}{f(a, b)} e^{ax+by}$.

In such a case, we write

$$\frac{1}{f(D, D')}\, e^{ax+by} = \frac{1}{f(D, D')}\, (e^{ax+by} \,.\, 1) = e^{ax+by}\, \frac{1}{f(D+a, D+b)} \cdot 1.$$

SOLVED EXAMPLES

Example 6. *Find the general solution of the following partial differential equations:*

(*i*) $(D^2 - 4DD' + 4D'^2 + D - 2D')\, z = e^{x+y}$

(*ii*) $(D + D' - 1)(D + D' - 3)(D + D')\, z = e^{x+y} \sin(2x + y)$.

Sol. (*i*) We have $\quad (D^2 - 4DD' + 4D'^2 + D - 2D')\, z = e^{x+y}$. ...(1)

$$D^2 - 4DD' + 4D'^2 + D - 2D' = (D - 2D')^2 + (D - 2D')$$

$$= (D - 2D')(D - 2D' + 1) = (1.D - 2D' + 0)(1.D - 2D' + 1)$$

$$\therefore \qquad \text{C.F.} = e^{-0.x/1}\, \phi_1(1.y - (-2)x) + e^{-1.x/1}\, \phi_2(1.y - (-2)x)$$

$$= \phi_1(y + 2x) + e^{-x}\, \phi_2(y + 2x).$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D - 2D' + 1)}\, e^{x+y} \qquad \text{...(2)}$$

Here $\quad a = 1, b = 1$.

$\therefore \quad D = 1, D' = 1 \Rightarrow D - 2D' = 1 - 2(1) = -1 \neq 0$ and $D - 2D' + 1 = 1 - 2(1) + 1 = 0$

$$\therefore \qquad \text{P.I.} = \frac{1}{D - 2D' + 1}\left(\frac{1}{D - 2D'}\, e^{x+y}\right) = \frac{1}{D - 2D' + 1}\left(\frac{1}{1 - 2(1)}\, e^{x+y}\right)$$

$$= \frac{1}{D - 2D' + 1} \cdot \frac{e^{x+y}}{-1} = -\, e^{x+y}\, \frac{1}{(D+1) - 2(D'+1) + 1} \cdot 1$$

$$= -\, e^{x+y}\, \frac{1}{D - 2D'}\, 1 = -\, e^{x+y} \,.\, \frac{1}{D}\left(1 - 2\frac{D'}{D}\right)^{-1} \cdot 1$$

$$= -\, e^{x+y} \,.\, \frac{1}{D}\left(1 + 2\frac{D'}{D} + \ldots\ldots\right).1 = -\, e^{x+y} \,.\, \frac{1}{D}\,(1) = -\, x e^{x+y}.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$\mathbf{z = \phi_1(y + 2x) + e^{-x}\, \phi_2(y + 2x) - xe^{x+y}}$, where ϕ_1, ϕ_2 are arbitrary functions.

(*ii*) We have $(D + D' - 1)(D + D' - 3)(D + D')\, z = e^{x+y} \sin(2x + y)$. ...(1)

$\Rightarrow$ $(1.D + 1.D' - 1)(1.D + 1.D' - 3)(1.D + 1.D' + 0)z = e^{x+y} \sin(2x + y)$

$\therefore$ C.F. $= e^{-(-1)x/1}\, \phi_1(1.y - 1.x) + e^{-(-3)x/1}\, \phi_2(1 \cdot y - 1 \cdot x) + e^{-0.x/1}\, \phi_3(1 \cdot y - 1 \cdot x)$

$= e^x\, \phi_1(y - x) + e^{3x}\, \phi_2(y - x) + \phi_3(y - x)$.

$$\text{P.I.} = \frac{1}{(D + D' - 1)(D + D' - 3)(D + D')}\, e^{x+y} \sin(2x + y)$$

$$= e^{x+y} \frac{1}{(D + 1 + D' + 1 - 1)\,(D + 1 + D' + 1 - 3)\,(D + 1 + D' + 1)} \sin(2x + y)$$

$$= e^{x+y} \frac{1}{(D + D' + 1)\,(D + D' - 1)\,(D + D' + 2)} \sin(2x + y)$$

$$= e^{x+y} \frac{1.(D + D' - 2)}{((D + D')^2 - 1)\,((D + D')^2 - 4)} \sin(2x + y)$$

$$= e^{x+y} \frac{D + D' - 2}{(D^2 + D'^2 + 2DD' - 1)\,(D^2 + D'^2 + 2DD' - 4)} \sin(2x + y)$$

$$= e^{x+y} \frac{D + D' - 2}{(-4 - 1 + 2(-1).2.1 - 1)\,(-4 - 1 + 2(-1)(2.1) - 4)} \sin(2x + y)$$

$$= e^{x+y} \frac{D + D' - 2}{-130} \sin(2x + y)$$

$$= -\frac{1}{130}\, e^{x+y}\, (2\cos(2x + y) + \cos(2x + y) - 2\sin(2x + y))$$

$$= -\frac{1}{130}\, e^{x+y}\, (3\cos(2x + y) - 2\sin(2x + y)).$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of (1) is

$$\mathbf{z = e^x\, \phi_1(y - x) + e^{3x}\, \phi_2(y - x) + \phi_3(y - x)}$$

$$\mathbf{- \frac{1}{130}\, e^{x+y}\, (3\cos(2x + y) - 2\sin(2x + y)),}$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

EXERCISE E

Find the general solution of the following partial differential equations :

1. $(D^2 - DD' + D' - 1)\, z = e^y$

2. $(D^2 - DD' + D' - 1)\, z = e^x$

3. $(D^2 - D')\, z = e^{x+y}$

4. $(D^2 - D'^2 - 3D + 3D')\, z = xy + e^{x+2y}$

5. $(D^2 + DD' + D + D' - 1)\, z = e^{-2x}\, (x^2 + y^2)$

6. $D(D - 2D')(D + D')\, z = e^{x+2y}\, (x^2 + 4y^2)$.

NOTES

Answers

1. $z = e^x \phi_1(y) + e^{-x} \phi_2(y + x) - xe^y$

2. $z = e^x \phi_1(y) + e^{-x} \phi_2(y + x) + \frac{1}{2} xe^x$

3. $z = \sum_{i=1}^{\infty} c_i e^{a_i x + a_i^2 y} - ye^{x+y}$

4. $z = \phi_1(y + x) + e^{3x} \phi_2(y - x) - \frac{1}{6} x^2 y - \frac{1}{9} xy - \frac{1}{9} x^2 - \frac{1}{18} x^3 - \frac{2}{27} x - xe^{x+2y}$

5. $z = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y} + e^{-2x} (x^2 + y^2 + 6x + 2y + 18)$, where $a^2_i + a_i b_i + a_i + b_i - 1 = 0$

6. $z = \phi_1(y) + \phi_2(y + 2x) + \phi_3(y - x) - \frac{1}{81} (9x^2 + 36y^2 - 18x - 72y + 76) e^{x+2y}$.

6. PARTIAL DIFFERENTIAL EQUATIONS REDUCIBLE TO EQUATIONS WITH CONSTANT COEFFICIENTS

STRUCTURE

INTRODUCTION

Till now we have been discussing the solution of linear partial differential equations which are with constant coefficients. Now we shall consider the method of solving a particular type of linear partial differential equations with variable coefficients that are capable of reducing to a linear partial differential equations with constant coefficients.

REDUCIBLE LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Let $$f(x\mathrm{D}, y\mathrm{D}') = \mathrm{F}(x, y) \quad \ldots(1)$$

be a linear partial differential equation with variable coefficients. Here $f(x\mathrm{D}, y\mathrm{D}')$ is some function of $x\mathrm{D}$ and $y\mathrm{D}'$ such that (1) may be a linear partial differential equation. The following are some of the partial differential equations of the form $f(x\mathrm{D}, y\mathrm{D}') = \mathrm{F}(x, y)$:

(*i*) $x^2\mathrm{D}^2 - y^2\mathrm{D}'^2 = x^2y$

(*ii*) $(x^2\mathrm{D}^2 - 2xy\mathrm{DD}' + y^2\mathrm{D}'^2 - x\mathrm{D} + 3y\mathrm{D}')\, z = 8(y/x)$

(*iii*) $x^2r - y^2t + px - qy = \log x$.

In general, a reducible linear partial differential equation is of the form :

$$a_0x^n \frac{\partial^n z}{\partial x^n} + a_1x^{n-1}y \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + a_n y^n \frac{\partial^n z}{\partial y^n} + \ldots\ldots = \mathrm{F}(x, y)$$

This equation can also be written as

$$(a_0x^n\mathrm{D}^n + a_1x^{n-1}y\mathrm{D}^{n-1}\mathrm{D}' + \ldots\ldots + a_n y^n \mathrm{D}'^n + \ldots\ldots)\, z = \mathrm{F}(x, y).$$

SOLUTION OF REDUCIBLE LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Let $f(xD, yD') = (a_0 x^n D^n + a_1 x^{n-1} y D^{n-1} D' + \ldots\ldots + a_n y^n D'^n + \ldots\ldots)\, z = F(x, y)$ ...(1)

be a reducible linear partial differential equation with variable coefficients.

Define variables u and v by $u = \log x$ and $v = \log y$.

$\therefore \quad x = e^u$ and $y = e^v$.

Let $D_1 = \dfrac{\partial}{\partial u}$ and $D'_1 = \dfrac{\partial}{\partial v}$.

Now $$x\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = x\frac{\partial z}{\partial u}\frac{1}{x} = \frac{\partial z}{\partial u}$$

$$\therefore \quad x\frac{\partial}{\partial x} = \frac{\partial}{\partial u} \quad \text{or} \quad xD = D_1 \qquad \ldots(2)$$

Also, $$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = x\left[x^{n-1}\frac{\partial^n z}{\partial x^n} + (n-1)\,x^{n-2}\frac{\partial^{n-1} z}{\partial x^{n-1}}\right]$$

$$= x^n\frac{\partial^n z}{\partial x^n} + (n-1)\,x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\therefore \quad x^n\frac{\partial^n z}{\partial x^n} = \left(x\frac{\partial}{\partial x} - (n-1)\right)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\therefore \quad x^n\frac{\partial^n}{\partial x^n} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1}}{\partial x^{n-1}}$$

or $$x^n D^n = (xD - n + 1)\, x^{n-1} D^{n-1}$$

or $$x^n D^n = (D_1 - n + 1)\, x^{n-1} D^{n-1}. \qquad \ldots(3)$$

When $n = 2$, (3) $\Rightarrow$ $x^2D^2 = (D_1 - 1)\, xD = (D_1 - 1)D_1$

$\therefore \quad x^2D^2 = D_1(D_1 - 1)$

When $n = 3$, (3) $\Rightarrow$ $x^3D^3 = (D_1 - 2)\, x^2D^2 = (D_1 - 2)\, D_1(D_1 - 1)$

$\therefore \quad x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$ etc.

Thus, we have $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$, $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$, ...

Similarly, we have $yD' = D_1'$, $y^2D'^2 = D_1'(D_1' - 1)$, $y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2)$ ......

Also, $xyDD' = D_1D_1'$,

and $x^r y^s D^r D'^s = D_1(D_1 - 1) \ldots\ldots (D_1 - r + 1)D_1'(D_1' - 1) \ldots\ldots (D_1' - s + 1)$.

Substituting these values, the given equation will be reduced to a linear partial differential equation with constant coefficients and with independent variables u and v. This equation is solved by known methods and then the values of u and v are substituted in terms x and y. This represents the general solution of the given equation.

SOLVED EXAMPLES

Example 1. *Find the general solution of the following partial differential equations :*

(*i*) $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} + x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} - z = 0$

(*ii*) $x^2 \dfrac{\partial^2 z}{\partial x^2} - 4xy \dfrac{\partial^2 z}{\partial x \partial y} + 4y^2 \dfrac{\partial^2 z}{\partial y^2} + 6y \dfrac{\partial z}{\partial y} = x^3 y^4$

(*iii*) $x^2 r - y^2 t + px - qy = \log x$

(*iv*) $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} + nz = n\left(x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y}\right) + x^2 + y^2 + x^3$.

Sol. (*i*) We have $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} + x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} - z = 0$.

$\Rightarrow \quad (x^2D^2 + 2xyDD' + y^2D'^2 + xD + yD' - 1)\, z = 0 \quad \ldots(1)$

Let $u = \log x$ and $v = \log y$.

$\therefore$ $x = e^u$ and $y = e^v$.

Also, $xD = D_1$, $yD' = D_1'$, $x^2D^2 = D_1(D_1 - 1)$, $xyDD' = D_1D_1'$, $y^2D'^2 = D_1'(D_1' - 1)$,

where $D_1 = \dfrac{\partial}{\partial u}$ and $D_1' = \dfrac{\partial}{\partial v}$.

$\therefore$ (1) $\Rightarrow$ $(D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1) + D_1 + D_1' - 1)\, z = 0$

$\Rightarrow \quad (D_1'^2 + 2D_1D_1' + D_1'^2 - 1)\, z = 0$

$\Rightarrow \quad ((D_1 + D_1')^2 - 1)\, z = 0$

$\Rightarrow \quad (D_1 + D_1' - 1)(D_1 + D_1' + 1)\, z = 0$

$\Rightarrow \quad (1.D_1 + 1.D_1' - 1)(1.D_1 + 1.D_1' + 1)\, z = 0.$

$\therefore \quad z = e^{-(-1).u/1}\, \phi_1(1.v - 1.u) + e^{-1.u/1}\, \phi_2(1.v - 1.u)$

$= e^u\, \phi_1(v - u) + e^{-u}\, \phi_2(v - u)$

$= x\phi_1(\log y - \log x) + \dfrac{1}{x}\, \phi_2(\log y - \log x)$

$= x\phi_1\left(\log \dfrac{y}{x}\right) + \dfrac{1}{x}\phi_2\left(\log \dfrac{y}{x}\right) = x\psi_1\left(\dfrac{y}{x}\right) + \dfrac{1}{x}\psi_2\left(\dfrac{y}{x}\right)$, (say)

$\therefore$ The general solution is $\mathbf{z = x\psi_1\left(\dfrac{y}{x}\right) + \dfrac{1}{x}\psi_2\left(\dfrac{y}{x}\right)}$, where ψ_1, ψ_2 are arbitrary functions.

(*ii*) We have $x^2 \dfrac{\partial^2 z}{\partial x^2} - 4xy \dfrac{\partial^2 z}{\partial x \partial y} + 4y^2 \dfrac{\partial^2 z}{\partial y^2} + 6y \dfrac{\partial z}{\partial y} = x^3 y^4$.

$\Rightarrow \quad (x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')\, z = x^3y^4 \quad \ldots(1)$

Let $u = \log x$ and $v = \log y$.

$\therefore$ $x = e^u$ and $y = e^v$.

Also, $xD = D_1$, $yD' = D_1'$, $x^2D^2 = D_1(D_1 - 1)$, $xyDD' = D_1D_1'$, $y^2D_1'^2 = D_1'(D_1' - 1)$,

where $D_1 = \dfrac{\partial}{\partial u}$ and $D_1' = \dfrac{\partial}{\partial v}$.

$\therefore$ (1) $\Rightarrow$ $(D_1(D_1-1) - 4D_1D_1' + 4D_1'(D_1'-1) + 6D_1')\, z = e^{3u}e^{4v}$

$\Rightarrow$ $(D_1^2 - 4D_1D_1' + 4D_1'^2 - D_1 + 2D_1')\, z = e^{3u+4v}$

$\Rightarrow$ $((D_1 - 2D_1')^2 - (D_1 - 2D_1'))\, z = e^{3u+4v}$

$\Rightarrow$ $(D_1 - 2D_1')(D_1 - 2D_1' - 1)\, z = e^{3u+4v}$

$(1.D_1 - 2D_1' + 0)(1.D_1 + (-2)D_1' - 1)\, z = e^{3u+4v}$

$\therefore$ C.F. $= e^{0.u/1}\,\phi_1(1.v + 2u) + e^{1.u/1}\,\phi_2(1.v + 2u)$

$= \phi_1(2u + v) + e^u\,\phi_2(2u + v) = \phi_1(2\log x + \log y) + x\phi_2(2\log x + \log y)$

$= \phi_1(\log x^2y) + x\phi_2(\log x^2y)$

$= \psi_1(x^2y) + x\psi_2(x^2y)$, say

$$\text{P.I.} = \frac{1}{(D_1 - 2D_1')(D_1 - 2D_1' - 1)}\, e^{3u+4v} \qquad ...(2)$$

Here $a = 3,\ b = 4$

$D_1 = 3,\ D_1' = 4 \ \Rightarrow\ D_1 - 2D_1' = 3 - 2(4) = -5 \neq 0$

and $D_1 - 2D_1' - 1 = 3 - 2(4) - 1 = -6 \neq 0$

$\therefore$ (2) $\Rightarrow$ $\text{P.I.} = \dfrac{1}{(-5)(-6)}\, e^{3u+4v} = \dfrac{1}{30} x^3 y^4$.

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \psi_1(x^2y) + x\psi_2(x^2y) + \frac{1}{30}x^3y^4}$, where ψ_1, ψ_2 are arbitrary functions.

(*iii*) We have $x^2r - y^2t + px - qy = \log x$.

$\Rightarrow$ $(x^2D^2 - y^2D'^2 + xD - yD')\, z = \log x$ ...(1)

Let $u = \log x$ and $v = \log y$.

$\therefore$ $x = e^u$ and $y = e^v$

Also, $xD = D_1,\ yD' = D_1',\ x^2D^2 = D_1(D_1 - 1),\quad y^2D'^2 = D_1'(D_1' - 1)$,

where $D_1 = \dfrac{\partial}{\partial u}$ and $D_1' = \dfrac{\partial}{\partial v}$.

$\therefore$ (1) $\Rightarrow$ $(D_1(D_1 - 1) - D_1'(D_1' - 1) + D_1 - D_1')\, z = u$

$\Rightarrow$ $(D_1^2 - D_1'^2)\, z = u \ \Rightarrow\ (D_1 + D_1')(D_1 - D_1')\, z = u$

$\Rightarrow$ $(1.D_1 + 1.D_1' + 0)(1.D_1 + (-1).D_1' + 0)z = u$

$\therefore$ C.F. $= e^{-0.u/1}\,\phi_1(1.v - 1\,.\,u) + e^{-0.u/1}\,\phi_2(1.v - (-1).u)$

$= \phi_1(v - u) + \phi_2(v + u)$

$= \phi_1(\log y - \log x) + \phi_2(\log y + \log x)$

$= \phi_1\left(\log \dfrac{y}{x}\right) + \phi_2(\log xy) = \psi_1\left(\dfrac{y}{x}\right) + \psi_2(xy)$, (say.)

$$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}\, u = \frac{1}{D_1^2}\left(1 - \frac{D_1'^2}{D_1^2}\right)^{-1} u$$

$$= \frac{1}{D_1^2}\left(1 + \frac{D_1'^2}{D_1^2} - ......\right) u = \frac{1}{D_1^2}(u + 0) = \frac{u^3}{6} = \frac{1}{6}(\log x)^3.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$\mathbf{z = \psi_1\left(\frac{y}{x}\right) + \psi_2(yx) + \frac{1}{6}(\log x)^3}$, where ψ_1, ψ_2 are arbitrary functions.

(*iv*) We have $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} + nz = n\left(x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y}\right) + x^2 + y^2 + x^3$.

$\Rightarrow \quad (x^2D^2 + 2xyDD' + y^2D'^2 - nxD - nyD' + n)\, z = x^2 + y^2 + x^3 \qquad \ldots(1)$

Let $\quad u = \log x$ and $v = \log y$.

$\therefore \quad x = e^u$ and $y = e^v$.

Also, $\quad xD = D_1, \quad yD' = D_1', \quad x^2D^2 = D_1(D_1 - 1), \; xyDD' = D_1D_1'$,

$$y^2D'^2 = D_1'(D_1' - 1), \text{ where } D_1 = \frac{\partial}{\partial u} \text{ and } D_1' = \frac{\partial}{\partial v}.$$

$(1) \Rightarrow (D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1) - nD_1 - nD_1' + n)\, z = e^{2u} + e^{2v} + e^{3u}$

$\Rightarrow \quad (D_1^2 + 2D_1D_1' + D_1'^2 - D_1 - D_1' - nD_1 - nD_1' + n)\, z = e^{2u} + e^{2v} + e^{3u}$

$\Rightarrow \quad ((D_1 + D_1')^2 - (n + 1)(D_1 + D_1') + n)\, z = e^{2u} + e^{2v} + e^{3u}$

$\Rightarrow \quad (D_1 + D_1' - 1)(D_1 + D_1' - n)\, z = e^{2u} + e^{2v} + e^{3u}$

$\Rightarrow \quad (1.D_1 + 1.D_1' - 1)(1.D_1 + 1.D_1' - n)z = e^{2u} + e^{2v} + e^{3u}$.

$\therefore$ C.F. $= e^{1.u/1}\, \phi_1(1.v - 1.u) + e^{nu/1}\, \phi_2(1.v - 1.u)$

$$= e^u\, \phi_1(v - u) + e^{nu}\, \phi_2(v - u) = x\phi_1\left(\log \frac{y}{x}\right) + x^n \phi_2\left(\log \frac{y}{x}\right)$$

$$= x\psi_1(y/x) + x^n\psi_2(y/x), \quad \text{(say.)}$$

$$\text{P.I.} = \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}(e^{2u} + e^{2v} + e^{3u})$$

$$= \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{2u} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{2v} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{3u}$$

$$= \frac{1}{(2 + 0 - 1)(2 - 0 - n)} e^{2u} + \frac{1}{(0 + 2 - 1)(0 + 2 - n)} e^{2v} + \frac{1}{(3 + 0 - 1)(3 + 0 - n)} e^{3u}$$

$$= \frac{1}{2 - n} x^2 + \frac{1}{2 - n} y^2 + \frac{1}{2(3 - n)} x^3.$$

$\therefore$ Using G.S. = C.F. + P.I., the general solution of the given equation is

$$\mathbf{z = x\psi_1\left(\frac{y}{x}\right) + x^n\psi_2\left(\frac{y}{x}\right) + \frac{1}{2 - n}(x^2 + y^2) + \frac{1}{2(3 - n)} x^3},$$

where ψ_1 and ψ_2 are arbitrary functions.

WORKING STEPS OF SOLVING f(xD, yD′) = F(x, y)

Step I. *Put $u = \log x$ and $v = \log y$.*

Step II. *Change the whole equation in independent variables u and v* by using $x = e^u$ and $y = e^v$. *We shall get a linear partial differential equation with constant coefficients.*

Step III. *Find the general solution of the equation obtained in* **step II.**

Step IV. *In the general solution, put $u = \log x$ and $v = \log y$. This gives the general solution of the given equation.*

NOTES

EXERCISE

Find the general solution of the following partial differential equations :

1. $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$
2. $x^2 \dfrac{\partial^2 z}{\partial x^2} - 3xy \dfrac{\partial^2 z}{\partial x \partial y} + 2y^2 \dfrac{\partial^2 z}{\partial y^2} + 5y \dfrac{\partial z}{\partial y} - 2z = 0$
3. $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} + x \dfrac{\partial z}{\partial x} - y \dfrac{\partial z}{\partial y} = 0$
4. $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = xy$
5. $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = x^2 y$
6. $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = x^m y^n$, $m + n \neq 0, 1$
7. $x^2 r - 3xys + 2y^2 t + px + 2qy = x + 2y$
8. $x^2 r + 2xys - xp = \dfrac{x^3}{y^2}$
9. $yt - q = xy$
10. $x^2 r + 2xys + y^2 t = (x^2 + y^2)^{n/2}$
11. $x^2 r - xys - 2y^2 t + xp - 2yq = \log \dfrac{y}{x} - \dfrac{1}{2}$
12. $x^2 r - 4y^2 t - 4yq - z = x^2 y^2 \log y$.

Answers

1. $z = \psi_1\left(\dfrac{y}{x}\right) + x\psi_2\left(\dfrac{y}{x}\right)$
2. $z = x^2 \psi_1 (xy) + \dfrac{1}{x} \psi_2 (x^2 y)$
3. $z = \psi_1(xy) + \psi_2\left(\dfrac{y}{x}\right)$
4. $z = \psi_1(xy) + x\psi_2\left(\dfrac{y}{x}\right) + xy \log x$
5. $z = \psi_1(xy) + x\psi_2\left(\dfrac{y}{x}\right) + \dfrac{1}{2} x^2 y$
6. $z = \psi_1\left(\dfrac{y}{x}\right) + x\psi_2\left(\dfrac{y}{x}\right) + \dfrac{x^m y^n}{(m+n)(m+n-1)}$
7. $z = \psi_1(xy) + \psi_2(x^2 y) + x + y$
8. $z = \psi_1(y) + x^2\psi_2\left(\dfrac{y}{x^2}\right) - \dfrac{x^3}{9y^2}$
9. $z = \psi_1(x) + y^2\psi_2(x) + \dfrac{1}{2} xy^2 \log y$
10. $z = \psi_1\left(\dfrac{y}{x}\right) + x\psi_2\left(\dfrac{y}{x}\right) + \dfrac{(x^2 + y^2)^{n/2}}{n(n-1)}$
11. $z = \psi_1(x^2 y) + \psi_2\left(\dfrac{y}{x}\right) + \dfrac{1}{2} (\log x)^2 \log y - \dfrac{1}{4} (\log x)^2$
12. $z = \sum_{i=1}^{\infty} c_i x^{a_i} y^{b_i} + x^2 y^2 \dfrac{(16 - 15 \log y)}{225}$, where $a_i^2 - 4b_i^2 - a_i - 1 = 0$.

UNIT III

7. MONGE'S METHODS

NOTES

STRUCTURE

INTRODUCTION

In the last chapter we discussed the methods of solving some special type of linear partial differential equations with variable coefficients which were capable of being reduced to linear partial differential equations with constant coefficients by changing the independent variables. Solving any given partial differential equation with variable coefficients is not an easy task. We are moving in this direction step by step.

PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER

A partial differential equation of the second order is of the form $f(x, y, z, p, q, r, s, t) = 0$. It is only in special cases that a partial differential equation of second order can be solved. Monge's methods are used to solve some particular types of equations of second order.

INTERMEDIATE INTEGRAL

Let
$$f(x, y, z, p, q, r, s, t) = 0 \qquad ...(1)$$
be a partial differential equation of second order. A relation of the form $u = \phi(v)$, where u, v are functions of x, y, z, p, q and ϕ is an arbitrary function, is called an **intermediate integral** of (1) if the given partial differential equation (1) could be derived by eliminating the arbitrary function ϕ.

For example $p - \frac{1}{x} \log y = \phi(x)$ is an intermediate integral of the equation $xys = 1$, because differentiating $p - \frac{1}{x} \log y = \phi(x)$ w.r.t. y, we get $\frac{\partial p}{\partial y} - \frac{1}{x} . \frac{1}{y} = 0$ or $s - \frac{1}{xy} = 0$ or $xys = 1$.

Remark. Finding of one or more intermediate integrals of a partial differential equation of second order is the first step in the direction of finding the general solution of the given partial differential equation of second order.

NOTES

MONGE'S METHODS

Let $$\text{R}r + \text{S}s + \text{T}t + \text{U}(rt - s^2) = \text{V} \qquad ...(1)$$

be a partial differential equation of second order, where R, S, T, U, V are functions of x, y, z, p, q. An equation of the form (1) may or may not admit of a solution. Monge's methods are used to solve any solvable equation of the form (1).

In particular if U = 0, then (1) reduces to $\text{R}r + \text{S}s + \text{T}t = \text{V}$. We shall be considering the cases U = 0 and U ≠ 0 separately.

MONGE'S METHOD OF SOLVING Rr + Ss + Tt = V

Let $$\text{R}r + \text{S}s + \text{T}t = \text{V} \qquad ...(1)$$

be a solvable partial differential equation, where R, S, T, V are functions of x, y, z, p, q.

Since z is a function of x and y, we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy = r\,dx + s\,dy$$

and $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy = s\,dx + t\,dy.$$

Solving these equations for r and t, we get $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$.

$$\therefore \quad (1) \quad \Rightarrow \quad \text{R}\left(\frac{dp - sdy}{dx}\right) + \text{S}s + \text{T}\left(\frac{dq - sdx}{dy}\right) = \text{V}$$

$$\Rightarrow \quad s[\text{R}(dy)^2 - \text{S}\,dxdy + \text{T}(dx)^2] = \text{R}\,dydp + \text{T}\,dxdq - \text{V}\,dxdy \qquad ...(2)$$

The equations : $$\text{R}(dy)^2 - \text{S}\,dxdy + \text{T}(dx)^2 = 0 \qquad ...(3)$$

and $$\text{R}\,dydp + \text{T}\,dxdq - \text{V}\,dxdy = 0 \qquad ...(4)$$

are called **Monge's equations.** The equation (3) may have either distinct or same factors.

Case I. Let $\text{R}(dy)^2 - \text{S}\,dxdy + \text{T}(dx)^2 = (\text{A}_1dy + \text{B}_1dx)(\text{A}_2dy + \text{B}_2dx) = 0.$

In this case we have two distinct systems

$$\left.\begin{aligned} \text{A}_1dy + \text{B}_1dx = 0 \\ \text{R}dydp + \text{T}dxdq - \text{V}dxdy = 0 \end{aligned}\right] \qquad ...(5)$$

and $$\left.\begin{aligned} \text{A}_2dy + \text{B}_2dx = 0 \\ \text{R}dydp + \text{T}dxdq - \text{V}dxdy = 0 \end{aligned}\right] \qquad ...(6)$$

Let system (5) be integrable. Let $u = u(x, y, z, p, q) = a$ and $v = v(x, y, z, p, q) = b$ satisfy the system (5).

$\therefore \quad u = \psi(v)$ is an intermediate integral of (1), since $u = a$, $v = b$ satisfy (2) and hence (1). If system (6) is also integrable, we get another intermediate integral. These intermediate integrals are solved to find the values of p and q in terms of x and y. The values of p and q are substituted in $dz = p\,dx + q\,dy$. This is integrated to get the general solution of (1). In case we get only one intermediate integral or we want to use only one intermediate integral then we express it in the form $Pp + Qq = R$ and use Lagrange's method to find the general solution of (1).

Case II. Let $R(dy)^2 - Sdx\,dy + T(dx)^2 = (Ady + Bdx)^2 = 0$

Let $u = u(x, y, z, p, q) = a$, $v = v(x, y, z, p, q) = b$ satisfy the system

$$Ady + Bdx = 0$$

$$Rdydp + Tdxdq - Vdxdy = 0$$

NOTES

$\therefore$ $u = \psi(v)$ is an intermediate integral of (1), since $u = a$, $v = b$ satisfy (2) and hence (1). We express it in the form $Pp + Qq = R$ and use Lagrange's method to find the general solution of (1).

Type I. Equations giving two distinct intermediate integrals and both are used to find the general solution.

WORKING STEPS FOR SOLVING PROBLEMS	
Step I.	*Write the given equation in the form* $Rr + Ss + Tt = V$.
Step II.	*Substitute the values of R, S, T, V in the Monge's equations :* $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ...(1) *and* $Rdydp + Tdxdq - Vdxdy = 0$ ...(2)
Step III.	*Factorise (1) into two distinct factors.*
Step IV.	*Find two intermediate integrals. Solve these to find the values of p and q.*
Step V.	*Put p and q in* $dz = pdx + qdy$ *and integrate to get the general solution of the given equation.*

SOLVED EXAMPLES

Example 1. *Find the general solution of the following partial differential equations :*

(*i*) $r - t\cos^2 x + p\tan x = 0$ (*ii*) $xy(r - t) - s(x^2 - y^2) = qx - py$.

Sol. (*i*) We have $r - t\cos^2 x + p\tan x = 0$.

$\Rightarrow$ $$r - t\cos^2 x = -p\tan x \quad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we get

$$R = 1,\ S = 0,\ T = -\cos^2 x,\ V = -p\tan x.$$

The **Monge's equations** are :

$$R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad ...(2)$$

and $$Rdydp + Tdxdq - Vdxdy = 0 \quad ...(3)$$

(2) $\Rightarrow$ $$(dy)^2 - \cos^2\alpha\,(dx)^2 = 0$$

$\Rightarrow$ $$(dy - \cos\alpha\,dx)(dy + \cos\alpha\,dx) = 0$$

$\Rightarrow$ $$dy - \cos x\,dx = 0 \quad ...(4)$$

and $$dy + \cos\alpha\,dx = 0 \quad ...(4')$$

(3) $\Rightarrow$ $$1.dydp + (-\cos^2\alpha)\,dxdq - (-p\tan x)\,dxdy = 0$$

$\Rightarrow$ $$dydp - \cos^2\alpha\,dxdq + p\tan x\,dx\,dy = 0 \quad ...(5)$$

We consider the system (4) and (5).

Integrating (4), we get $y - \sin x = a$

NOTES

(4) $\Rightarrow$ $dy = \cos x\, dx$. Putting this value of dy in (5), we get

$$\cos x\, dxdp - \cos^2 x\, dxdq + p \tan x\, dx \cos x\, dx = 0$$

$$\Rightarrow \quad \cos x\, dx\, (dp - \cos x\, dq + p \tan x\, dx) = 0$$

$$\Rightarrow \quad dp - \cos x\, dq + p \tan x\, dx = 0$$

$$\Rightarrow \quad (dp + p \tan x\, dx) - \cos x\, dq = 0$$

$\Rightarrow$ Multiplying by $\sec x$, we get

$$(\sec x\, dp + p \tan x \sec x\, dx) - dq = 0$$

$$\Rightarrow \quad d(p \sec x) - dq = 0$$

Integrating, we get $\quad p \sec x - q = b$

Let $\quad b = \phi(a)$, ϕ arbitrary.

$\therefore \quad p \sec x - q = \phi(y - \sin x)$ ...(6)

Similarly by solving (4′) and (5), we get $\quad p \sec x + q = \psi(y + \sin x)$ ...(7)

Solving (6) and (7), we get

$$p = \frac{1}{2 \sec x} (\phi(y - \sin x) + \psi(y + \sin x))$$

and $$q = -\frac{1}{2} (\phi(y - \sin x) - \psi(y + \sin x))$$

Now $\quad dz = pdx + qdy.$

$$\therefore \quad dz = \frac{\cos x}{2} (\phi(y - \sin x) + \psi\, (y + \sin x)dx$$

$$-\frac{1}{2} (\phi\, (y - \sin x) - \psi\, (y + \sin x))\, dy$$

$$\Rightarrow \quad dz = \frac{1}{2} \phi(y - \sin x)\, (\cos x\, dx - dy) + \frac{1}{2} \psi\, (y + \sin x)\, (\cos x\, dx + dy)$$

$$\Rightarrow \quad dz = -\frac{1}{2} \phi(y - \sin x)\, d(y - \sin x) + \frac{1}{2} \psi(y + \sin x)\, d(y + \sin x)$$

Integrating, we get

$$z = \int -\frac{1}{2} \phi(y - \sin x)\, d(y - \sin x) + \int \frac{1}{2} \psi(y + \sin x)\, d(y + \sin x)$$

$$\Rightarrow \quad \mathbf{z = \phi_1(y - \sin x) + \phi_2(y + \sin x),} \quad \textbf{(say.)}$$

This is the general solution of the given equation. Here ϕ_1, ϕ_2 are arbitrary functions.

(*ii*) We have $\quad xy(r - t) - s(x^2 - y^2) = qx - py.$

$\Rightarrow \quad xyr - (x^2 - y^2)\, s - xyt = qx - py$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we get

$$R = xy,\ S = -x^2 + y^2,\ T = -xy,\ V = qx - py.$$

The **Monge's equations** are :

$R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ...(2)

and $\quad Rdydp + Tdxdq - Vdxdy = 0$ ...(3)

(2) $\Rightarrow \quad xy(dy)^2 + (x^2 - y^2)\, dxdy - xy(dx)^2 = 0$

$\Rightarrow \quad (xdy - ydx)(ydy + xdx) = 0$

$\Rightarrow \quad xdy - ydx = 0$ ...(4)

and $\quad ydy + xdx = 0$ ...(4′)

(3) $\Rightarrow \quad xydydp - xy\, dxdq - (qx - py)\, dxdy = 0$ ...(5)

We consider the system (4) and (5),

(4) $\Rightarrow$ $xdy = ydx \Rightarrow \dfrac{dy}{y} = \dfrac{dx}{x}$

Integrating, we get $\log y = \log x + \log a$

$\Rightarrow$ $\log \dfrac{y}{x} = \log a \Rightarrow \dfrac{y}{x} = a.$

NOTES

(5) $\Rightarrow$ $ydp \,.\, xdy - xdq \,.\, ydx - qdx \,.\, xdy + pdy \,.\, ydx = 0$

$\Rightarrow$ $ydp - xdq - qdx + pdy = 0$ [By using (4)]

$\Rightarrow$ $(ydp + pdy) - (xdq + qdx) = 0$

$\Rightarrow$ $d(yp - xq) = 0 \Rightarrow yp - xq = b$

Let $b = \phi(a)$. $\therefore$ $yp - xq = \phi(y/x)$ ...(6)

Now we consider the system (4′) and (5).

(4′) $\Rightarrow$ $d(x^2 + y^2) = 0 \Rightarrow x^2 + y^2 = c$

(5) $\Rightarrow$ $xdp \,.\, ydy - ydq \,.\, xdx - qdy \,.\, xdx + pdx \,.\, ydy = 0$

$\Rightarrow$ $xdp + ydq + qdy + pdx = 0$ [By using (4′)]

$\Rightarrow$ $(xdp + pdx) + (ydq + qdy) = 0$

$\Rightarrow$ $d(xp) + d(yq) = 0 \Rightarrow d(xp + yq) = 0 \Rightarrow xp + yq = k.$

Let $k = \psi(c)$. $\therefore$ $xp + yq = \psi(x^2 + y^2)$ ...(7)

$\therefore$ We get two intermediate integrals of (1).

Solving (6) and (7) for p and q, we get

$$p = \frac{x\psi(x^2 + y^2) + y\phi\,(y/x)}{x^2 + y^2} \quad \text{and} \quad q = \frac{y\psi(x^2 + y^2) - x\phi\,(y/x)}{x^2 + y^2}.$$

Now $dz = pdx + qdy$

$$\therefore \quad dz = \frac{x\psi(x^2 + y^2) + y\phi\,(y/x)}{x^2 + y^2}\,dx + \frac{y\psi(x^2 + y^2) - x\phi\,(y/x)}{x^2 + y^2}\,dy.$$

$$\Rightarrow \quad dz = \frac{\psi(x^2 + y^2)}{x^2 + y^2}(xdx + ydy) + \frac{\phi(y/x)}{x^2 + y^2}(ydx - xdy)$$

$$\Rightarrow \quad dz = \frac{\psi(x^2 + y^2)}{2\,(x^2 + y^2)}\,d(x^2 + y^2) - \frac{\phi(y/x)}{1 + (y/x)^2}\,\frac{xdy - ydx}{x^2}$$

$$\Rightarrow \quad dz = \frac{\psi(x^2 + y^2)}{2\,(x^2 + y^2)}\,d(x^2 + y^2) - \frac{\phi(y/x)}{1 + (y/x)^2}\,d\,(y/x)$$

Integrating, we get

$$z = \int \frac{\psi(x^2 + y^2)}{2\,(x^2 + y^2)}\,d(x^2 + y^2) + \int -\frac{\phi\,(y/x)}{1 + (y/x)^2}\,d\,(y/x).$$

$\Rightarrow$ $\mathbf{z = \phi_1(x^2 + y^2) + \phi_2(y/x),}$ **(say).**

This is the general solution of the given equation. Here ϕ_1, ϕ_2 are arbitrary functions.

Type II. Equations giving two distinct intermediate integral and only one is used to find the general solution.

WORKING STEPS FOR SOLVING PROBLEMS

Step I. *Write the given equation in the form* $Rr + Ss + Tt = V$.

Step II. *Substitute the values of R, S, T, V in the Monge's equations :*

$$R(dy)^2 - Sdxdy + T(dy)^2 = 0 \quad ...(1)$$

and $$Rdydp + Tdxdq - Vdxdy = 0 \quad ...(2)$$

Step III. *Factorize (1) into two distinct factors.*

Step IV. *Using any of the factors, find an intermediate integral. Solve this integral by using Lagrange method to get the general solution of the given equation.*

Example 2. *Find the general solution of the following partial differential equations :*

(*i*) $(r - s)y + (s - t)x + q - p = 0$ (*ii*) $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$.

Sol. (*i*) We have $(r - s)y + (s - t)x + q - p = 0$.

$$\Rightarrow \quad yr + (x - y)s - xt = p - q \quad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we get

$$R = y, \quad S = x - y, \quad T = -x, \quad V = p - q.$$

The **Monge's equations** are :

$$R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad ...(2)$$

and $$Rdydp + Tdxdq - Vdxdy = 0 \quad ...(3)$$

(2) $\Rightarrow \quad y(dy)^2 - (x - y)\, dxdy - x(dx)^2 = 0$

$\Rightarrow \quad (dx + dy)(ydy - xdx) = 0$

$\Rightarrow \quad dx + dy = 0 \quad ...(4) \quad$ and $\quad ydx - xdy = 0 \quad ...(4')$

(3) $\Rightarrow \quad ydydp - xdx\, dq - (p - q)\, dx\, dy = 0 \quad ...(5)$

We consider the system (4) and (5).

Integrating (4), we get $x + y = a$.

(5) $\Rightarrow \quad ydydp - xdxdq - pdxdy + qdxdy = 0$

$\Rightarrow \quad ydpdy + xdq(-dx) + pdy(-dx) + qdxdy = 0$

$\Rightarrow \quad ydp + xdq + pdy + qdx = 0 \quad (\because\ dy = -dx)$

$\Rightarrow \quad (ydp + pdy) + (xdq + qdx) = 0$

$\Rightarrow \quad d(py) + d(qx) = 0 \quad \Rightarrow \quad py + qx = b$

Let $b = \psi(a)$. $\quad \therefore \quad py + qx = \psi(x + y)$

This is a Lagrange linear equation.

Auxiliary equations are $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{\psi(x + y)} \quad ...(6)$

(6) $\Rightarrow \quad \dfrac{dx}{y} = \dfrac{dy}{x} \quad \Rightarrow \quad xdx - ydy = 0 \quad \Rightarrow \quad \dfrac{1}{2} d(x^2 - y^2) = 0$

$\Rightarrow \quad x^2 - y^2 = k$, where k is arbitrary.

(6) $\Rightarrow \quad \dfrac{dx}{y} = \dfrac{dz}{\psi(x + y)} \quad \Rightarrow \quad \dfrac{dx}{\sqrt{x^2 - k}} = \dfrac{dz}{\psi\left(x + \sqrt{x^2 - k}\right)}$

NOTES

$$\Rightarrow \qquad dz = \frac{\psi\left(x+\sqrt{x^2-k}\right)}{\sqrt{x^2-k}}\, dx \qquad \text{...(7)}$$

$$u = x + \sqrt{x^2-k} \quad \Rightarrow \quad du = \left(1+\frac{x}{\sqrt{x^2-k}}\right) dx = \frac{u}{\sqrt{x^2-k}}\, dx \quad \Rightarrow \quad \frac{dx}{\sqrt{x^2-k}} = \frac{du}{u}$$

$\therefore$ (7) $\Rightarrow$ $dz = \psi(u) \, . \, \frac{du}{u} \quad \Rightarrow \quad dz = \frac{\psi(u)}{u}\, du$

Integrating, we get

$$z = \int \frac{\psi(u)}{u}\, du + b.$$

$$\Rightarrow \qquad z = \phi_1(u) + b, \quad \text{say}$$

$$\Rightarrow \qquad z = \phi_1(x+\sqrt{x^2-k}) + \phi_2\,(a), \quad \text{say}$$

$$\Rightarrow \qquad \mathbf{z = \phi_1\,(x+y) + \phi_2\,(x^2-y^2).}$$

This is the general solution of the given equation. Here ϕ_1, ϕ_2 are arbitrary functions.

(*ii*) We have $(x-y)\,(xr - xs - ys + yt) = (x+y)\,(p-q)$.

$\Rightarrow \quad (x^2 - xy)\, r - (x^2 - y^2)\, s + (xy - y^2)\, t = (x+y)\,(p-q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we get

$$R = x^2 - xy, \quad S = -(x^2-y^2), \quad T = xy - y^2, \quad V = (x+y)\,(p-q).$$

The **Monge's equations** are :

$$R(dy)^2 - S dxdy + T(dx)^2 = 0 \qquad \text{...(2)}$$

and

$$R dydp + T dxdq - V dxdy = 0 \qquad \text{...(3)}$$

(2) $\Rightarrow$ $(x^2 - xy)(dy)^2 + (x^2-y^2)\, dxdy + (xy-y^2)(dx)^2 = 0$

$\Rightarrow$ $x(dy)^2 + (x+y)dxdy + y(dx)^2 = 0$

$\Rightarrow$ $(xdy + ydx)(dx+dy) = 0$

$\Rightarrow$ $xdy + ydx = 0$ ...(4) and $dx + dy = 0$ ...(4′)

(3) $\Rightarrow$ $(x^2-xy)\, dydp + (xy - y^2)\, dxdq - (x+y)(y+p)\, dxdy = 0$

We consider the system (4) and (5).

(4) $\Rightarrow$ $d(xy) = 0 \quad \Rightarrow \quad xy = a.$

(5) $\Rightarrow$ $(x-y)dp \, . \, xdy + (x-y)dq \, . \, ydx - (p-q)dx \, . \, xdy - (p-q)dy \, . \, ydx = 0$

$\Rightarrow$ $(x-y)dp - (x-y)dq - (p-q)dx + (p-q)dy = 0$ $\quad (\because \ xdy = -ydx)$

$\Rightarrow$ $(x-y)(dp-dq) - (p-q)(dx-dy) = 0$

$$\Rightarrow \qquad \frac{dp-dq}{p-q} = \frac{dx-dy}{x-y} \quad \Rightarrow \quad \frac{d(p-q)}{p-q} = \frac{d\,(x-y)}{x-y}$$

Integrating, let $\quad p - q = b(x-y).$

Let $\ b = \psi(a). \quad \therefore \quad p - q = (x-y)\,\psi(xy).$

This is a Lagrange linear equation.

Auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{(x-y)\,\psi(xy)}$ ...(6)

(6) $\Rightarrow$ $dx = -dy \quad \Rightarrow \quad x + y = c.$

NOTES

Taking $y\psi(xy)$, $x\psi(xy)$, 1 as multipliers, each fraction of (6)

$$= \frac{y\psi(xy)\,dx + x\psi(xy)\,dy + dz}{y\psi(xy) - x\psi(xy) + (x-y)\,\psi(xy)} = \frac{y\psi(xy)\,dx + x\psi(xy)\,dy + dz}{0}.$$

$\therefore \quad y\psi(xy)\,dx + x\psi(xy)\,dy + dz = 0$

$\Rightarrow \quad \psi(xy)(ydx + xdy) + dz = 0 \quad \Rightarrow \quad \psi(xy)\;d(xy) + dz = 0$

Integrating, let $\phi_1(xy) + z = \lambda$

Let $\lambda = \phi_2(c)$. $\quad \therefore \quad \phi_1(xy) + z = \phi_2(x + y)$

$\therefore$ The general solution of the given equation is

$\mathbf{z = \phi_2(x + y) - \phi_1(xy)}$, where ϕ_1 and ϕ_2 are arbitrary functions.

Type III. Equations giving two identical intermediate integrals.

WORKING STEPS FOR SOLVING PROBLEMS

Step I. *Write the given equation in the form $Rr + Ss + Tt = V$.*

Step II. *Substitute the values of R, S, T, V in the Monge's equations :*

$$R(dy)^2 - Sdxdy + T(dy)^2 = 0 \quad ...(1)$$

and $\quad Rdydp + Tdxdq - Vdxdy = 0 \quad ...(2)$

Step III. *Factorise (1) into two identical factors.*

Step IV. *Using one factor, find an intermediate integral. Solve this integral by using Lagrange method to get the general solution of the given equation.*

Example 3. *Find the general solution of the following partial differential equations :*

(*i*) $y^2r - 2ys + t = p + 6y$ $\qquad$ (*ii*) $(y - x)\,(q^2r - 2pqs + p^2t) = (p + q)^2\,(p - q)$.

Sol. (*i*) We have $\quad y^2r - 2ys + t = p + 6y$. $\quad$ ...(1)

Comparing (1) with Rr + Ss + Tt = V, we get

$$R = y^2, \quad S = -2y, \quad T = 1, \quad V = p + 6y.$$

The **Monge's equations** are $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ $\quad$ ...(2)

and $\quad Rdydp + Tdxdq - Vdxdy = 0$ $\quad$ ...(3)

(2) $\Rightarrow \quad y^2(dy)^2 + 2y\,dxdy + (dx)^2 = 0$

$\Rightarrow \quad (ydy + dx)^2 = 0 \quad \Rightarrow \quad ydy + dx = 0 \quad \Rightarrow \quad dx = -\,ydy$ $\quad$ ...(4)

Integrating, let $x = -\dfrac{y^2}{2} + a \quad \Rightarrow \quad y^2 + 2x = b$, where $b = 2a$

(3) $\Rightarrow \quad y^2dydp + 1.dxdq - (p + 6y)\,dxdy = 0$ $\quad$ ...(5)

$\Rightarrow \quad y^2dydp - ydydq + y(p + 6y)\,(dy)^2 = 0$

$\Rightarrow \quad ydp - dq + (p + 6y)\,dy = 0$

$\Rightarrow \quad (ydp + pdy) - dq + 6ydy = 0$

Integrating, $\quad py - q + 3y^2 = k$.

Let $\quad k = \psi(b)$. $\quad \therefore \quad py - q + 3y^2 = \psi(y^2 + 2x)$

This is a Lagrange linear equation.

The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\psi(y^2 + 2x) - 3y^2} \quad ...(6)$$

NOTES

(6) $\Rightarrow$ $\dfrac{dx}{y} = \dfrac{dy}{-1} \Rightarrow ydy + dx = 0 \Rightarrow y^2 + 2x = c$

(6) $\Rightarrow$ $-dy = \dfrac{dz}{\psi(c) - 3y^2} \Rightarrow (3y^2 - \psi(c))\, dy = dz$

Integrating, let $y^3 - y\psi(c) = z + d$.

Let $d = \phi_1(c)$

$\therefore$ $\mathbf{y^3 - y\psi(y^2 + 2x) = z + \phi_1(y^2 + 2x).}$

This is the general solution of (1). Here ψ, ϕ_1 are arbitrary functions.

(*ii*) We have $(y - x)(q^2r - 2pqs + p^2t) = (p + q)^2\,(p - q)$.

$\Rightarrow$ $(y - x)q^2r - 2pq(y - x)s + p^2(y - x)t = (p + q)^2\,(p - q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we get

$$R = (y - x)q^2,\; S = -2pq(y - x),\; T = p^2(y - x),\; V = (p + q)^2\,(p - q).$$

The **Monge's equations** are $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ...(2)

and $Rdydp + Tdxdq - Vdxdy = 0$ ...(3)

(2) $\Rightarrow$ $(y - x)\, q^2(dy)^2 + 2pq(y - x)dxdy + p^2(y - x)(dx)^2 = 0$

$\Rightarrow$ $(y - x)(qdy + pdx)^2 = 0 \Rightarrow pdx + qdy = 0 \Rightarrow dz = 0 \Rightarrow z = a.$

(3) $\Rightarrow$ $(y - x)\, q^2dydp + p^2(y - x)\, dxdq - (p + q)^2\,(p - q)\, dxdy = 0$

$\Rightarrow$ $(y - x)\,[qdp\,.\,qdy + pdq\,.\,pdx] - (p^2 - q^2)\,[pdx\,.\,dy + qdy\,.\,dx] = 0$

$\Rightarrow$ $(y - x)\,(qdp - pdq) - (p^2 - q^2)\,(-dy + dx) = 0$

($\because$ $pdx + qdy = 0$)

$\Rightarrow$ $qdp - pdq - (p^2 - q^2)\,\dfrac{d(x - y)}{y - x} = 0$

$\Rightarrow$ $q^2d\left(\dfrac{p}{q}\right) + (p^2 - q^2)\,\dfrac{d(x - y)}{x - y} = 0$

$\Rightarrow$ $\dfrac{d(x - y)}{x - y} + \dfrac{1}{(p/q)^2 - 1}\, d\left(\dfrac{p}{q}\right) = 0$

Integrating, we get $\log\,(x - y) + \dfrac{1}{2}\log\dfrac{p/q - 1}{p/q + 1} = \dfrac{1}{2}\log b$.

$\Rightarrow$ $(x - y)^2\,\dfrac{p - q}{p + q} = b$

Let $b = \psi(a)$. $\therefore$ $(x - y)^2\,\dfrac{p - q}{p + q} = \psi(z)$

$\Rightarrow$ $(x - y)^2\,(p - q) - (p + q)\,\psi(z) = 0$

$\Rightarrow$ $p((x - y)^2 - \phi(z)) - q((x - y)^2 + \psi(z)) = 0$ ...(4)

This is a Lagrange linear equation. The auxiliary equations are

$$\frac{dx}{(x - y)^2 - \phi(z)} = \frac{dy}{-((x - y)^2 + \psi(z))} = \frac{dz}{0} \qquad \text{...(5)}$$

(5) $\Rightarrow$ $dz = 0 \Rightarrow z = c.$

Each fraction of (5) $= \dfrac{dx + dy}{-2\psi(z)} = \dfrac{dx - dy}{2(x - y)^2}$

$\Rightarrow$ $d(x + y) = -\psi(c)\,\dfrac{d(x - y)}{(x - y)^2}$

Integrating, we get $$x + y = -\psi(c) \cdot \frac{(x-y)^{-1}}{-1} + d.$$

$$\Rightarrow \quad x + y - \frac{\psi(z)}{x-y} = d$$

Let $d = \phi(c)$.

$$\therefore \quad \mathbf{x + y - \frac{\psi(z)}{x-y} = \phi(z)}.$$

This represents the general solution of (1). Here ψ, ϕ are arbitrary functions.

EXERCISE A

Find the general solution of the following partial differential equations by using Monge's method :

1. $r = k^2t$
2. $t - r\sec^4 y = 2q\tan y$
3. $(r - s)x = (t - s)y$
4. $pt - qs = q^3$
5. $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0$
6. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$
7. $x^2r - y^2t - 2xp + 2z = 0$
8. $x^2r - 2xs + t + q = 0$
9. $y^2r + 2xys + x^2t + px + qy = 0$

Answers

1. $z = \phi_1(y - kx) + \phi_2(y + kx)$
2. $z = \phi_1(\tan y - x) + \phi_2(\tan y + x)$
3. $\dfrac{z}{x+y} = \phi_1(x + y) + \phi_2(y/x)$
4. $y = xz + \phi_1(z) + \phi_2(x)$
5. $x = \phi_1(z) + \phi_2(x + y + z)$
6. $z = xy + \phi_1(x^2 + y^2) + \phi_2(x/y)$
7. $zy = (xy)^{3/2}\,\phi_1(y/x) + \phi_2(xy)$
8. $z = x\phi_1(y + \log x) + \phi_2(y + \log x)$
9. $z = \psi\,(y^2 - x^2)\log(y + x) + \phi\,(y^2 - x^2)$.

Hint

3. The intermediate integrals are $p - q = f(y/x)$ and $xp + yq - z = g(x + y)$.
Solving for p and q, we get

$$p = \frac{1}{x+y}\left[z + g(x+y) + yf\left(\frac{y}{x}\right)\right]$$

and $$q = \frac{1}{x+y}\left[z + g(x+y) - xf\left(\frac{y}{x}\right)\right].$$

$$\therefore \quad dz = \frac{1}{x+y}\left[z + g(x+y) + yf\left(\frac{y}{x}\right)\right]dx + \frac{1}{x+y}\left[z + g(x+y) - xf\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \quad (x+y)\,dz = zd(x+y) + g(x+y)\,d(x+y) + f\left(\frac{y}{x}\right)(ydx - xdy)$$

$$\Rightarrow \quad \frac{(x+y)\,dz - zd(x+y)}{(x+y)^2} = \frac{g(x+y)}{(x+y)^2}\,d(x+y) + \frac{f(y/x)}{1+(y/x)^2}\,d(y/x).$$

MONGE'S METHOD OF SOLVING $Rr + Ss + Tt + U(rt - s^2) = V$

NOTES

Let $$Rr + Ss + Tt + U(rt - s^2) = V \quad ...(1)$$
be a solvable partial differential equation, where R, S, T, U, V are functions of x, y, z, p, q.

Since z is a function of x and y, we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy = rdx + sdy$$

and $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy = sdx + tdy.$$

Solving these equations for r and t, we get $r = \dfrac{dp - sdy}{dx}$ and $t = \dfrac{dq - sdx}{dy}$.

$$\therefore \quad (1) \Rightarrow \quad R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right)$$
$$+ U\left(\frac{(dp - sdy)(dq - sdx)}{dxdy} - s^2\right) = V$$

$$\Rightarrow \quad s[R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq)]$$
$$= Rdydp + Tdxdq + Udpdq - Vdxdy \quad ...(2)$$

The equations : $R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq) = 0 \quad ...(3)$

and $$Rdydp + Tdxdq + Udpdq - Vdxdy = 0 \quad ...(4)$$

are called **Monge's equations.** Here, the equation (3) cannot be factored.

Let $\lambda = \lambda(x, y, z, p, q)$ be a function such that

$$\lambda [R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq)] + Rdydp + Tdxdq + Udpdq - Vdxdy$$

be factorisable.

Let $$\lambda [R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq)] + Rdydp + Tdxdq + Udpdq - Vdxdy$$
$$= (ady + bdx + cdp)(\alpha dy + \beta dx + \gamma dq)$$
$$= a\alpha(dy)^2 + (a\beta + b\alpha)\, dxdy + b\beta(dx)^2 + c\beta dxdp + a\gamma dydq + c\alpha dydp + b\gamma dxdq + c\gamma dpdq = 0$$

Comparing coefficients, we get

$$a\alpha = \lambda R,\ a\beta + b\alpha = -\lambda S - V,\ b\beta = \lambda T,\ c\beta = \lambda U,\ a\gamma = \lambda U,$$
$$c\alpha = R,\ b\gamma = T,\ c\gamma = U.$$

Let $a = \lambda,\ \alpha = R. \quad \therefore \quad a\alpha = \lambda R$

Also $a\gamma = \lambda U \Rightarrow \gamma = U,\ c\alpha = R \Rightarrow cR = R \Rightarrow c = 1,$

$c\beta = \lambda U \Rightarrow 1\,.\,\beta = \lambda U \Rightarrow \beta = \lambda U,\ b\gamma = T \Rightarrow bU = T \Rightarrow b = T/U.$

$\therefore \quad a\beta + b\alpha = -\lambda S - V \Rightarrow \lambda(\lambda U) + (T/U)\, R = -\lambda S - V \quad ...(5)$

$\Rightarrow \quad U^2\lambda^2 + SU\lambda + TR + UV = 0$

Let λ_1, λ_2 be the roots of (5).

We have $(ady + bdx + cdp)(\alpha dy + \beta dx + \gamma dq) = 0 \quad ...(6)$

NOTES

Taking $\lambda = \lambda_1$, (6) becomes $\left(\lambda_1 dy + \frac{T}{U} dx + 1 . dp\right)(Rdy + \lambda_1 Udx + Udq) = 0$

$\Rightarrow \quad (\lambda_1 Udy + Tdx + Udp)\,(Rdy + \lambda_1 Udx + Udq) = 0 \qquad ...(7)$

Similarly, taking $\lambda = \lambda_2$, we get $(\lambda_2 Udy + Tdx + Udp)\,(Rdy + \lambda_2 Udx + Udq) = 0 \qquad ...(8)$

Equations (7) and (8) give four systems of equations:

$$\left.\begin{array}{l}\lambda_1 Udy + Tdx + Udp = 0 \\ \lambda_2 Udy + Tdx + Udp = 0\end{array}\right] \qquad ...(9)$$

$$\left.\begin{array}{l}\lambda_1 Udy + Tdx + Udp = 0 \\ Rdy + \lambda_2 Udx + Udq = 0\end{array}\right] \qquad ...(10)$$

$$\left.\begin{array}{l}\lambda_2 Udy + Tdx + Udp = 0 \\ Rdy + \lambda_1 Udx + Udq = 0\end{array}\right] \qquad ...(11)$$

$$\left.\begin{array}{l}Rdy + \lambda_1 Udx + Udq = 0 \\ Rdy + \lambda_2 Udx + Udq = 0\end{array}\right] \qquad ...(12)$$

Subtracting equations of (9), we get $(\lambda_1 - \lambda_2)\, Udy = 0$.

If $\lambda_1 \neq \lambda_2$, then $Udy = 0$ identically, which is not true.

$\therefore$ System (9) does not give intermediate integral.

Similarly, we reject system (12).

$\therefore$ We have only two systems given below:

$$\left.\begin{array}{l}\lambda_1 Udy + Tdx + Udp = 0 \\ Rdy + \lambda_2 Udx + Udq = 0\end{array}\right] \qquad ...(10)$$

$$\left.\begin{array}{l}\lambda_2 Udy + Tdx + Udp = 0 \\ Rdy + \lambda_1 Udx + Udq = 0\end{array}\right] \qquad ...(11)$$

The equation (5) may have either distinct or same roots.

Case I. $\lambda_1 \neq \lambda_2$

In this case we have two distinct systems (10) and (11). Let system (10) be integrable. Let $u = u(x, y, z, p, q) = a$, $v = v(x, y, z, p, q) = b$ satisfy the system (10).

$\therefore$ $u = \psi(v)$ is an intermediate integral of (1).

If system (11) is also integrable, we get another intermediate integral. These intermediate integrals are solved to find the values of p and q in terms of x and y. The values of p and q are substituted in $dz = pdx + qdy$. This is integrated to get the general solution of (1). In case we get only one intermediate integral or we want to use only one intermediate integral then we express it in the form $Pp + Qq = R$ and use Lagrange's method to find the solution of (1).

Case II. $\lambda_1 = \lambda_2$

In this case we have identical systems (10) and (11). Let system (10) be integrable. Let $u = u(x, y, z, p, q) = a$, $v = v(x, y, z, p, q) = b$ satisfy the system (10).

$\therefore$ $u = \psi(v)$ is an intermediate integral of (1). We express it in the form $Pp + Qq = R$ and use Lagrange's method to find the solution of (1).

Remark. In case the computation of finding general solution of an equation is difficult, we restrict ourselves to a solution with arbitrary constants.

NOTES

SOLVED EXAMPLES

Example 4. *Find the solution of the following partial differential equations :*

(*i*) $r + 4s + t + rt - s^2 = 2$ (*ii*) $3r + s + t + rt - s^2 = -9.$

Sol. (*i*) We have $r + 4s + t + rt - s^2 = 2$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = 1,\ S = 4,\ T = 1,\ U = 1,\ V = 2.$$

λ-quadratic equation is $U^2\lambda^2 + SU\lambda + TR + UV = 0.$

$\Rightarrow$ $\lambda^2 + 4\lambda + (1 + 2) = 0 \Rightarrow \lambda = -1, -3.$

Let $\lambda_1 = -1, \lambda_2 = -3.$

First system of equations giving intermediate integral is

$$\lambda_1 U dy + T dx + U dp = 0 \quad ...(2)$$

$$R dy + \lambda_2 U dx + U dq = 0 \quad ...(3)$$

(2) $\Rightarrow$ $-dy + dx + dp = 0$ ...(4)

(3) $\Rightarrow$ $dy - 3dx + dq = 0$ ...(5)

Integrating (4) and (5), we get $-y + x + p = a$ and $y - 3x + q = b$, where a and b are arbitrary constants.

Let $b = \psi(a)$. $\therefore$ $y - 3x + q = \psi(-y + x + p)$

In particular let $\psi(-y + x + p) = \alpha(-y + x + p) + \beta.$

$\therefore$ $y - 3x + q = \alpha(-y + x + p) + \beta$

$\Rightarrow$ $\alpha p - q = -(\alpha + 3)x + (\alpha + 1)y - \beta$

This is a Lagrange equation.

The auxiliary equations are $\dfrac{dx}{\alpha} = \dfrac{dy}{-1} = \dfrac{dz}{-(\alpha+3)x + (\alpha+1)y - \beta}$ ...(6)

(6) $\Rightarrow$ $\dfrac{dx}{\alpha} = -dy \Rightarrow dx + \alpha dy = 0 \Rightarrow x + \alpha y = \gamma.$ ...(7)

Equating second and third fractions of (6), we get

$$\frac{dy}{-1} = \frac{dz}{-(\alpha+3)(\gamma - \alpha y) + (\alpha+1)y - \beta} \quad \text{(Using (7))}$$

$\Rightarrow$ $$dy = \frac{dz}{-(\alpha^2 + 4\alpha + 1)y + \alpha\gamma + 3\gamma + \beta}$$

$\Rightarrow$ $((\alpha^2 + 4\alpha + 1)y - (\alpha\gamma + 3\gamma + \beta))\,dy + dz = 0$

$\Rightarrow$ $(\alpha^2 + 4\alpha + 1)\dfrac{y^2}{2} - (\alpha\gamma + 3\gamma + \beta)y + z = k$

$\Rightarrow$ $(\alpha^2 + 4\alpha + 1)\dfrac{y^2}{2} - ((\alpha + 3)(x + \alpha y) + \beta)y + z = \phi(\gamma)$ [Taking $k = \phi(\gamma)$]

$\Rightarrow$ $\left(\dfrac{\alpha^2}{2} + \alpha - \dfrac{1}{2}\right)y^2 + (\alpha+3)xy + (\alpha+3)\beta y = z - \phi(x + \alpha y)$

$\Rightarrow$ $$\mathbf{z = \frac{1}{2}(\alpha^2 + 2\alpha - 1)\,y^2 + (\alpha + 3)\,xy + (\alpha + 3)\,\beta y + \phi(x + \alpha y).}$$

This is the general solution of the given equation.

Here α, β are arbitrary constants and ϕ is an arbitrary function.

Remark. For the above equation, $y - 3x + q = \psi(-y + x + p)$ is an intermediate integral. Since p appears in the argument of the arbitrary function ψ, we cannot find the value of p using this equation and the other intermediate integral of the given equation.

NOTES

(*ii*) We have $3r + s + t + rt - s^2 = -9$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = 3,\ S = 1,\ T = 1,\ U = 1,\ V = -9.$$

λ-quadratic equation is $U^2\lambda^2 + SU\lambda + TR + UV = 0$.

$\Rightarrow \quad \lambda^2 + \lambda - 6 = 0 \quad \Rightarrow \quad \lambda = 2, -3.$

Let $\quad \lambda_1 = 2, \lambda_2 = -3.$

First system of equations giving intermediate integral is

$$\lambda_1 U dy + T dx + U dp = 0 \quad ...(2)$$

$$R dy + \lambda_2 U dx + U dq = 0 \quad ...(3)$$

(2) $\Rightarrow \quad 2dy + dx + dp = 0 \quad \Rightarrow \quad 2y + x + p = a$

(3) $\Rightarrow \quad 3dy - 3dx + dq = 0 \quad \Rightarrow \quad 3y - 3x + q = b$

Let $a = \psi(b)$. $\quad \therefore \quad 2y + x + p = \psi(3y - 3x + q)$

In particular, let $\psi(3y - 3x + q) = \alpha(3y - 3x + q) + \beta$.

$\therefore \quad 2y + x + p = \alpha(3y - 3x + q) + \beta$

$\Rightarrow \quad p - \alpha q = (3\alpha - 2)y - (3\alpha + 1)x + \beta$

This is a Lagrange's equation. The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-\alpha} = \frac{dz}{(3\alpha - 2)\, y - (3\alpha + 1)\, x + \beta} \quad ...(4)$$

(4) $\Rightarrow \quad dx = \dfrac{dy}{-\alpha} \quad \Rightarrow \quad dy + \alpha dx = 0 \quad \Rightarrow \quad y + \alpha x = \gamma.$ ...(5)

(4) $\Rightarrow \quad dx = \dfrac{dz}{(3\alpha - 2)(\gamma - \alpha x) - (3\alpha + 1)x + \beta}$ (Using (5))

$\Rightarrow \quad (-(3\alpha^2 + \alpha + 1)\, x + 3\alpha\gamma - 2\gamma + \beta)\, dx = dz$

$\Rightarrow \quad z = -\dfrac{1}{2}(3\alpha^2 + \alpha + 1)\, x^2 + (3\alpha\gamma - 2\gamma + \beta)\, x + k$

$\Rightarrow \quad z = -\dfrac{1}{2}(3\alpha^2 + \alpha + 1)\, x^2 + (3\alpha y + 2\alpha^2 x - 2y - 2\alpha x + \beta)x + \phi(\gamma)$

[(Putting $k = \phi(\gamma)$]

$\Rightarrow \quad \mathbf{z = \dfrac{1}{2}(3\alpha^2 - 5\alpha - 1)\, x^2 + (3\alpha - 2)\, xy + \beta x + \phi(y + \alpha x).}$

This is the general solution of the given equation.

Here α and β are arbitrary constants and ϕ is an arbitrary function.

Example 5. *Find the solution of the following partial differential equations:*

(*i*) $5r + 6s + 3t + 2(rt - s^2) + 3 = 0$

(*ii*) $(q^2 - 1)\, zr - 2pqzs + (p^2 - 1)\, zt + z^2(rt - s^2) = p^2 + q^2 - 1.$

Sol. (*i*) We have $\quad 5r + 6s + 3t + 2(rt - s^2) = -3.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = 5,\ S = 6,\ T = 3,\ U = 2,\ V = -3.$$

λ-quadratic equation is $U^2\lambda^2 + SU\lambda + TR + UV = 0$.

$\Rightarrow \quad 4\lambda^2 + 12\lambda + 9 = 0 \quad \Rightarrow \quad \lambda = -3/2, -3/2.$

Let $\quad \lambda_1 = -3/2, \quad \lambda_2 = -3/2.$

$\therefore$ The system of equations giving intermediate integral is

$$\lambda_1 U dy + T dx + U dp = 0 \quad ...(2)$$

$$R dy + \lambda_2 U dx + U dq = 0 \quad ...(3)$$

(2) $\Rightarrow$ $-3dy + 3dx + 2dp = 0 \Rightarrow -3y + 3x + 2p = a$

(3) $\Rightarrow$ $5dy - 3dx + 2dq = 0 \Rightarrow 5y - 3x + 2q = b$

Let $a = \psi(b)$. $\therefore$ $-3y + 3x + 2p = \psi(5y - 3x + 2q)$

In particular, let $\psi(5y - 3x + 2q) = \alpha(5y - 3x + 2q) + \beta$.

$\therefore$ $-3y + 3x + 2p = \alpha(5y - 3x + 2q) + \beta$

$\Rightarrow$ $2p - 2\alpha q = 3y - 3x + 5\alpha y - 3\alpha x + \beta$.

This is a Lagrange's equation. The auxiliary equations are

$$\frac{dx}{2} = \frac{dy}{-2\alpha} = \frac{dz}{3y - 3x + 5\alpha y - 3\alpha x + \beta} \quad ...(4)$$

(4) $\Rightarrow$ $dx = -\dfrac{dy}{\alpha} \Rightarrow dy + \alpha dx = 0 \Rightarrow y + \alpha x = \gamma$. ...(5)

(4) $\Rightarrow$ $\dfrac{dx}{2} = \dfrac{dz}{(3 + 5\alpha)(\gamma - \alpha x) - 3(1+\alpha)x + \beta}$ (Using (5))

$\Rightarrow$ $(-(5\alpha^2 + 6\alpha + 3)x + 3\gamma + 5\alpha\gamma + \beta)\, dx = 2dz$

$\Rightarrow$ $-\dfrac{1}{2}(5\alpha^2 + 6\alpha + 3)x^2 + (3\gamma + 5\alpha\gamma + \beta)\, x = 2z + k$

$\Rightarrow$ $-\dfrac{1}{2}(5\alpha^2 + 6\alpha + 3)x^2 + ((3 + 5\alpha)(y + \alpha x) + \beta)\, x = 2z + \phi(\gamma)$ [Putting $k = \phi(\gamma)$]

$\Rightarrow$ $-\dfrac{1}{2}(5\alpha^2 + 6\alpha + 3)x^2 + (3\alpha x^2 + 5\alpha^2 x^2 + 3xy + 5\alpha xy + \beta x) = 2z + \phi(y + \alpha x)$

$\Rightarrow$ $\mathbf{\dfrac{1}{2}(5\alpha^2 - 3)\, x^2 + (3 + 5\alpha)\, xy + \beta x = 2z + \phi(y + \alpha x).}$

This is the general solution of the given equation.

Here α, β are arbitrary constants and ϕ is an arbitrary function.

(*ii*) We have $(q^2 - 1)\, zr - 2pqzs + (p^2 - 1)\, zt + z^2(rt - s^2) = p^2 + q^2 - 1$. ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$$R = (q^2 - 1)\, z,\ S = -2pqz,\ T = (p^2 - 1)\, z,\ U = z^2,\ V = p^2 + q^2 - 1.$$

λ-quadratic equation is $U^2\lambda^2 + SU\lambda + RT + UV = 0$.

$\Rightarrow$ $z^4\lambda^2 - 2pqz^3\lambda + (q^2 - 1)(p^2 - 1)\, z^2 + z^2\, (p^2 + q^2 - 1) = 0$

$\Rightarrow$ $z^4\lambda^2 - 2pqz^3\lambda + p^2q^2z^2 = 0 \Rightarrow (z\lambda - pq)^2 = 0 \Rightarrow \lambda = pq/z,\ pq/z.$

Let $\lambda_1 = pq/z,\ \lambda_2 = pq/z.$

$\therefore$ The system of equations giving intermediate integral is

$$\lambda_1 U dy + T dx + U dp = 0 \quad ...(2)$$

$$R dy + \lambda_2 U dx + U dq = 0 \quad ...(3)$$

(2) $\Rightarrow$ $\dfrac{pq}{z} . z^2 dy + (p^2 - 1)\, z\, dx + z^2 dp = 0$ ...(4)

(3) $\Rightarrow$ $(q^2 - 1)\, z dy + \dfrac{pq}{z} . z^2 dx + z^2 dq = 0$ ...(5)

(4) $\Rightarrow$ $pqdy + (p^2 - 1)\, dx + zdp = 0 \Rightarrow p(qdy + pdx) - dx + zdp = 0$

$\Rightarrow$ $pdz + zdp - dx = 0 \Rightarrow d(zp) - dx = 0 \Rightarrow zp - x = a.$

(5) $\Rightarrow$ $(q^2 - 1)\, dy + pqdx + zdq = 0 \Rightarrow q(qdy + pdx) - dy + zdq = 0$

$\Rightarrow$ $qdz + zdq - dy = 0 \Rightarrow d(zq) - dy = 0 \Rightarrow zq - y = b.$

Let $a = \psi(b)$ $\therefore$ $zp - x = \psi(zq - y)$.

NOTES

In particular, let $\psi(zq - y) = \alpha(zq - y) + \beta$.

$\therefore \quad zp - x = \alpha(zq - y) + \beta \Rightarrow zp - \alpha zq = x - \alpha y + \beta.$

This is a Lagrange's equation. The auxiliary equations are

$$\frac{dx}{z} = \frac{dy}{-\alpha z} = \frac{dz}{x - \alpha y + \beta} \qquad \text{...(6)}$$

$$\text{(6)} \Rightarrow \quad dx = -\frac{dy}{\alpha} \Rightarrow dy + \alpha dx = 0 \Rightarrow y + \alpha x = \gamma \qquad \text{...(7)}$$

$$\text{(6)} \Rightarrow \quad \frac{dx}{z} = \frac{dz}{x - \alpha(\gamma - \alpha x) + \beta}$$

$$\Rightarrow \quad [(1 + \alpha^2)x - \alpha\gamma + \beta]\, dx = zdz \qquad \text{(Using (7))}$$

$$\Rightarrow \quad (1 + \alpha^2)\frac{x^2}{2} - \alpha\gamma x + \beta x = \frac{z^2}{2} + k$$

$$\Rightarrow \quad (1 + \alpha^2)\, x^2 - 2x(y + \alpha x)\, x + 2\beta x = z^2 + 2k$$

$$\Rightarrow \quad (1 - \alpha^2)\, x^2 - 2\alpha xy + 2\beta x = z^2 + \phi(\gamma) \qquad \text{[Putting } 2k = \phi(\gamma)]$$

$$\Rightarrow \quad \mathbf{(1 - \alpha^2)\, x^2 - 2\alpha xy + 2\beta x = z^2 + \phi(y + \alpha x).}$$

This is the general solution of the given equation.

Here α, β are arbitrary constants and ϕ is an arbitrary function.

EXERCISE B

Find the solution of the following partial differential equations by using Monge's method:

1. $3s + (rt - s^2) = 2$

2. $3s - 2(rt - s^2) = 2$

3. $3r + 4s + t + (rt - s^2) = 1$

4. $2r - 6s + 2t + (rt - s^2) = 4$.

Answers

1. $z = \frac{\alpha}{2} y^2 + 2xy + \beta y - \phi(x + \alpha y)$

2. $\alpha z = \frac{5}{2} y^2 - 2\alpha xy - \beta y + \phi(\alpha x - 2y)$

3. $x^2 + 3y^2 + 2z - 4xy - 2\beta x = \phi\,(y + \alpha x)$

4. $z = (\alpha^2 + \alpha - 1)\, x^2 + (2\alpha - 2)\, xy + \beta x + \phi(y + \alpha x)$

UNIT IV

8. APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

STRUCTURE

INTRODUCTION

Many physical and engineering problems when formulated in the mathematical language give rise to partial differential equations. Besides these, partial differential equations also play an important role in the theory of Elasticity, Hydraulics etc.

Since, the general solution of a partial differential equation in a region R contains arbitrary constants or arbitrary functions, the unique solution of a partial differential equation corresponding to a physical problem will satisfy certain other conditions at the boundary of the region R. These are known as *boundary conditions.* When these conditions are specified for the time $t = 0$, they are known as *initial conditions.* A partial differential equation together with boundary conditions constitutes a *boundary value problem.*

In the applications of ordinary linear differential equations, we first find the general solution and then determine the arbitrary constants from the initial values. But the same method is not applicable to problems involving partial differential equations. Most of the boundary value problems involving linear partial differential equations can be solved by the method of separation of variables. In this method, right from the beginning, we try to find the particular solutions of the partial differential equation which satisfy all or some of the boundary conditions and then adjust them till the remaining conditions are also satisfied. A combination of these particular solutions gives the solution of the problem.

Fourier series is a powerful aid in determining the arbitrary functions.

NOTES

PRINCIPLE OF SUPERPOSITION

If $u_1, u_2, ..., u_n, ...$ are solutions of a homogeneous linear PDE in some region R, then

$$u = c_1 u_1 + c_2u_2 + \cdots + c_nu_n + \cdots = \sum_{n=1}^{\infty} c_n u_n$$

is also a solution of that PDE in the region R.

METHOD OF SEPARATION OF VARIABLES (or PRODUCT METHOD)

In this method, we assume the solution to be the product of two functions, each of which involves only one of the variables. The following examples explain the method.

SOLVED EXAMPLES

Example 1. *Solve by the method of separation of variables:* $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$

Sol. Here u is a function of x and y.

Let $\qquad u = X(x)\, Y(y) \qquad$ ...(1)

where X is a function of x only and Y is a function of y only, be a solution of the given equation.

Then $\qquad \dfrac{\partial u}{\partial x} = X'Y, \quad \dfrac{\partial u}{\partial y} = XY'$

Substituting in the given equation, we have

$$x^2X'Y + y^2XY' = 0$$

$$\Rightarrow \quad x^2X'Y = -y^2XY'$$

Separating the variables, we get

$$\frac{x^2X'}{X} = -\frac{y^2Y'}{Y} \qquad ...(2)$$

Since x and y are independent variables, as y varies x remains constant, so that the LHS and hence the RHS is constant. Therefore, equation (2) can hold only when each side is equal to the same constant, say k.

$$\therefore \quad \frac{x^2X'}{X} = -\frac{y^2Y'}{Y} = k$$

$\Rightarrow \quad \dfrac{X'}{X} = kx^{-2}$ and $\dfrac{Y'}{Y} = -ky^{-2}$ which are two ODEs.

Integrating, $\quad \log X = \dfrac{kx^{-1}}{-1} + \log c_1$ and $\log Y = -\dfrac{ky^{-1}}{-1} + \log c_2$

$\Rightarrow \quad \log \dfrac{X}{c_1} = -\dfrac{k}{x}$ and $\log \dfrac{Y}{c_2} = \dfrac{k}{y}$

$\Rightarrow$ $$X = c_1\, e^{-\frac{k}{x}} \quad \text{and} \quad Y = c_2\, e^{\frac{k}{y}}$$

Putting these values in (1), we have

$$u = c_1\, c_2\, e^{k\left(\frac{1}{y} - \frac{1}{x}\right)}$$

or $$u = ce^{k\left(\frac{1}{y} - \frac{1}{x}\right)} \qquad \text{where } c = c_1 c_2$$

is the required general solution.

Example 2. *Solve the equation* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$, *given that* $u(x, 0) = 6e^{-3x}$.

Sol. Here u is a function of x and t.

Let $$u = X(x)\, T(t) \qquad ...(1)$$

where X is a function of x only and T is a function of t only, be a solution of the given equation.

Then $$\frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = XT'$$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad X'T = (2T' + T)X$$

Separating the variables, we get $$\frac{X'}{X} = \frac{2T' + T}{T} \qquad ...(2)$$

Since x and t are independent variables, as t varies x remains constant, so that the LHS and hence the RHS is constant. Therefore, equation (2) can hold only when each side is equal to the same constant, say k.

$$\therefore \quad \frac{X'}{X} = k \quad i.e., \quad \log X = kx + \log c_1$$

or $$\log \frac{X}{c_1} = kx \quad \text{or} \quad X = c_1 e^{kx} \qquad ...(3)$$

and $$\frac{2T' + T}{T} = k \quad \text{or} \quad \frac{2T'}{T} + 1 = k \quad i.e., \quad \frac{T'}{T} = \frac{1}{2}(k-1)$$

or $$\log T = \frac{1}{2}(k-1)t + \log c_2 \quad \text{or} \quad \log\frac{T}{c_2} = \frac{1}{2}(k-1)t$$

or $$T = c_2\, e^{1/2\,(k-1)t} \qquad ...(4)$$

From (1), (3) and (4), we have $u = u(x, t) = c_1 e^{kx}.\ c_2 e^{1/2\,(k-1)t}$

$$= ce^{kx} \cdot e^{\frac{1}{2}(k-1)t} \qquad \text{where } c = c_1 c_2$$

Since, $$u(x, 0) = 6e^{-3x}$$

$$\therefore \quad ce^{kx} = 6e^{-3x} \qquad \text{[Given]}$$

$$\Rightarrow \quad c = 6 \quad \text{and} \quad k = -3$$

$\therefore$ The unique solution of the given equation is

$$u = 6e^{-3x}\,.\,e^{-2t} \quad i.e., \quad u = 6e^{-(3x + 2t)}.$$

Example 3. *Solve the equation* $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$ *by the method of separation of variables.*

NOTES

Sol. Here z is a function of x and y

Let $z = X(x)\,Y(y)$ ...(1)

where X is a function of X only and Y is a function of y only, be a solution of the given equation.

Then $\dfrac{\partial z}{\partial x} = X'Y,\ \dfrac{\partial z}{\partial y} = XY'$ and $\dfrac{\partial^2 z}{\partial x^2} = X''Y.$

Substituting in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \quad \text{or} \quad (X'' - 2X')Y + XY' = 0$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(2)

Since x and y are independent variables, equation (2) can hold only when each side is equal to the same constant, say k.

$$\therefore \quad \frac{X'' - 2X'}{X} = k \quad \text{or} \quad X'' - 2X' - kX = 0 \qquad ...(3)$$

and $$-\frac{Y'}{Y} = k \quad \text{or} \quad \frac{Y'}{Y} = -k \qquad ...(4)$$

These are ordinary differential equations.

For eqn. (3), the auxiliary equation is $m^2 - 2m - k = 0$ whose roots are $m = 1 \pm \sqrt{1+k}$.

$\therefore$ The solution of equation (3) is $X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}$

Also, the solution of (4) is $Y = c_3 e^{-ky}$

Substituting these values of X and Y in (1), the general solution of the given equation is

$$z = [c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}] \,.\, c_3 e^{-ky}$$

or $$z = [Ae^{(1+\sqrt{1+k})x} + Be^{(1-\sqrt{1+k})x}]\, e^{-ky},$$

where $A = c_1c_3$ and $B = c_2c_3$.

Example 4. *Use the method of separation of variables to solve the equation* $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u.$

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \qquad ...(1)$$

Let $u = X(x)\,Y(y)$ ...(2)

where, X is a function of x only and Y is a function of y only, be a solution of (1).

Then, $\dfrac{\partial u}{\partial y} = XY',\ \dfrac{\partial^2 u}{\partial x^2} = X''Y$

Substituting in (1), we have

$$X''Y = XY' + 2XY \quad \text{or} \quad X''Y = X(Y' + 2Y)$$

or $$\frac{X''}{X} = \frac{Y'}{Y} + 2 = -k^2 \quad \text{(say)}$$

(*i*) $$\frac{X''}{X} = -k^2 \Rightarrow \frac{d^2X}{dx^2} + k^2X = 0$$

A.E is $m^2 + k^2 = 0 \Rightarrow m = \pm ki$

Solution is $X = c_1 \cos kx + c_2 \sin kx$

(*ii*) $\dfrac{Y'}{Y} + 2 = -k^2$ or $\dfrac{Y'}{Y} = -(k^2 + 2)$

Integrating, $\log Y = -(k^2 + 2)\, y + \log c_3$

$\Rightarrow$ $Y = c_3\, e^{-(k^2+2)y}$

$\therefore$ From (2), the general solution is

$$u = (c_1 \cos kx + c_2 \sin kx)\, c_3\, e^{-(k^2+2)y}$$

or $u = (A \cos kx + B \sin kx)\, e^{-(k^2+2)y}$ where $A = c_1c_3$, $B = c_2c_3$.

Example 5. *Use the method of separation of variables to solve the equation* $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ *given that* $v = 0$ *when* $t \to \infty$ *as well as* $v = 0$ *at* $x = 0$ *and* $x = l$.

Sol. The given equation is $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ ...(1)

Let $v = X(x)\, T(t)$ ...(2)

where X is a function of x only and T is a function of t only be a solution of (1).

Then, $\dfrac{\partial^2 v}{\partial x^2} = X''T$ and $\dfrac{\partial v}{\partial t} = XT'$

Substituting in (1), we have

$$X''T = XT'$$

or $\dfrac{X''}{X} = \dfrac{T'}{T} = -k^2$ (say)

(*i*) $\dfrac{X''}{X} = -k^2 \Rightarrow X'' + k^2X = 0$

A.E is $m^2 + k^2 = 0$ so that $m = \pm ik$

Solution is $X = c_1 \cos kx + c_2 \sin kx$

(*ii*) $\dfrac{T'}{T} = -k^2$

Integrating, $\log T = -k^2t + \log c_3$

$\Rightarrow$ $T = c_3\, e^{-k^2t}$

$\therefore$ From (2) $v = (c_1 \cos kx + c_2 \sin kx)\, .\, c_3\, e^{-k^2t}$ ...(3)

Now, at $x = 0$, $v = 0$

$\therefore$ From (3), $0 = c_1\, c_3\, e^{-k^2t} \Rightarrow c_1 = 0$ ($\because$ $c_3 = 0$ would lead to trivial solution)

$\therefore$ (3) reduces to $v = c_2 \sin kx\, .\, c_3\, e^{-k^2t}$ ...(4)

At $x = l$, $v = 0$

$\therefore$ From (4), $0 = c_2c_3 \sin kl e^{-k^2t}$

Since $c_3 \neq 0$ and $c_2 = 0$ would lead to trivial solution.

$\therefore$ $\sin kl = 0 \Rightarrow kl = n\pi$

NOTES

$\Rightarrow \qquad k = \dfrac{n\pi}{l} \qquad$ where $n = 1, 2, 3, ...$

$\therefore$ From (4), $\qquad v = (c_2\, c_3) \sin \dfrac{n\pi x}{l}\, e^{-\frac{n^2\pi^2}{l^2}t}$

or $\qquad v_n = b_n \sin \dfrac{n\pi x}{l}\, e^{-\frac{n^2\pi^2}{l^2}t}$, where $c_2c_3 = b_n$

are the only possible solutions of (1) for $n = 1, 2, 3, ...$. Hence by superposition principle, the most general solution is

$$v = \sum_{n=1}^{\infty} b_n\, e^{-\left(\frac{n^2\pi^2}{l^2}\right)t} \sin \frac{n\pi x}{l}$$

Example 6. *Solve by the method of separation of variables:*

$$4\,\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \quad u = 3e^{-x} - e^{-5x}, \quad \text{when } t = 0.$$

Sol. Let $\qquad u = \text{XT} \qquad$ ...(1)

where X is a function of x only and T is a function of t only.

$$\therefore \qquad \frac{\partial u}{\partial t} = \text{XT}', \quad \frac{\partial u}{\partial x} = \text{TX}'$$

$\therefore$ From the given equation

$$4\text{XT}' + \text{TX}' = 3\text{XT}$$

or $$\frac{4\text{T}'}{\text{T}} + \frac{\text{X}'}{\text{X}} = 3$$

or $$\frac{4\text{T}'}{\text{T}} - 3 = -\frac{\text{X}'}{\text{X}} = p^2 \text{ (say)} \qquad ...(2)$$

(*i*) $$\frac{4\text{T}'}{\text{T}} = p^2 + 3$$

$$\frac{d\text{T}}{\text{T}} = \left(\frac{3 + p^2}{4}\right) dt$$

Integration yields,

$$\log \text{T} = \left(\frac{3 + p^2}{4}\right) t + \log c_1 \quad \Rightarrow \quad \text{T} = c_1\, e^{\left(\frac{3+p^2}{4}\right)t} \qquad ...(3)$$

(*ii*) $$\frac{-\text{X}'}{\text{X}} = p^2 \quad \Rightarrow \quad \frac{\text{X}'}{\text{X}} = -p^2 \quad \Rightarrow \quad \frac{d\text{X}}{\text{X}} = -p^2\, dx$$

Integration yields,

$$\log \text{X} = -p^2 x + \log c_2$$

$$\text{X} = c_2\, e^{-p^2 x} \qquad ...(4)$$

From (1), we get

$$u = \text{XT} = c_1 c_2\, e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

or $$u_n(x, t) = b_n\, e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t} \qquad \text{where} \quad c_1c_2 = b_n$$

Most general solution is

$$u\ (x,\ t) = \sum_{n=1}^{\infty} b_n\ e^{-p^2x + \left(\frac{3+p^2}{4}\right)t} \qquad ...(5)$$

when $t = 0$, $\qquad u(x,\ 0) = 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n\ e^{-p^2x}$

Comparing, when $\quad p^2 = 1,\ b_1 = 3 \quad$ and $\quad$ when $\quad p^2 = 5,\ b_2 = -1$

Hence, from (5), general solution is

$$u(x,\ t) = 3e^{-x+t} - e^{-5x+2t}$$

which is the required solution.

Example 7. *Solve the following equation by the method of separation of variables*

$$\frac{\partial^2 u}{\partial x\,\partial t} = e^{-t} \cos x$$

given that u = 0 when t = 0 and $\dfrac{\partial u}{\partial t}$ *= 0 when x = 0.*

Sol. Let $\qquad u = \text{XT} \qquad ...(1)$

where X is a function of x only and T is a function of t only.

Then, $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial t}(\text{XT}) = \text{X}\dfrac{d\text{T}}{dt}$

$$\therefore \qquad \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\text{X}\frac{d\text{T}}{dt}\right) = \frac{d\text{T}}{dt} \cdot \frac{d\text{X}}{dx} \qquad ...(2)$$

Substituting (2) in the given equation, we get

$$\frac{d\text{T}}{dt}\ \frac{d\text{X}}{dx} = e^{-t} \cos x$$

$$e^t\ \frac{d\text{T}}{dt} = \frac{\cos x}{\left(\dfrac{d\text{X}}{dx}\right)} = -p^2 \text{ (say)} \qquad ...(3)$$

Now, $\qquad e^t \dfrac{d\text{T}}{dt} = -p^2$

$\Rightarrow \qquad d\text{T} = -p^2 e^{-t}\ dt$

Integration yields,

$$\text{T} = p^2 e^{-t} + c_1 \qquad ...(4)$$

Also, $\qquad \dfrac{d\text{X}}{dx} = -\dfrac{1}{p^2}\ \cos x$

$$d\text{X} = -\frac{1}{p^2}\ \cos x\ dx$$

Integration yields,

$$\text{X} = -\frac{1}{p^2}\ \sin x + c_2 \qquad ...(5)$$

Using (4) and (5), we get from (1),

$$u(x,\ t) = \text{XT} = \left(-\frac{1}{p^2} \sin x + c_2\right)\ (p^2 e^{-t} + c_1) \qquad ...(6)$$

NOTES

Applying the condition $u = 0$ when $t = 0$ in (6), we get

$$0 = \left(-\frac{1}{p^2}\sin x + c_2\right)(p^2 + c_1)$$

$\Rightarrow$ $p^2 + c_1 = 0 \Rightarrow c_1 = -p^2$

From (6), $$\frac{\partial u}{\partial t} = \left(-\frac{1}{p^2}\sin x + c_2\right)(-p^2e^{-t}) \qquad ...(7)$$

Applying the condition $\frac{\partial u}{\partial t} = 0$ when $x = 0$ in (7), we get

$$0 = c_2(-p^2e^{-t})$$

$\Rightarrow$ $c_2 = 0$

Substituting the values of c_1 and c_2 in (6), we get

$$u(x, t) = -\frac{1}{p^2}\sin x\,(p^2e^{-t} - p^2)$$

$$= \sin x\,(1 - e^{-t})$$

Example 8. *Solve the P.D.E. by separation of variables method,*

$$u_{xx} = u_y + 2u,\ u(0, y) = 0,\ \frac{\partial}{\partial x}u(0, y) = 1 + e^{-3y}.$$

Sol. Let $u = XY$ ...(1)

where X is a function of x only and Y is a function of y only.

$\therefore$ $$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} = YX''$$

From the given equation,

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$\Rightarrow$ $$\frac{X''}{X} = \frac{Y'}{Y} + 2 = k\text{(say)} \qquad ...(2)$$

(*i*) $$\frac{X''}{X} = k$$

$\Rightarrow$ $X'' - kX = 0$

Auxiliary equation is

$$m^2 - k = 0$$

$\Rightarrow$ $m = \pm\sqrt{k}$

$\therefore$ $$\text{C.F.} = C_1e^{\sqrt{k}x} + C_2e^{-\sqrt{k}x}$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = C_1e^{\sqrt{k}x} + C_2e^{-\sqrt{k}x} \qquad ...(3)$$

(*ii*) $$\frac{Y'}{Y} + 2 = k$$

$$\Rightarrow \quad \frac{Y'}{Y} = k - 2$$

$$\Rightarrow \quad \frac{dY}{Y} = (k - 2)\, dy$$

Integration yields,

$$\log Y = (k - 2)y + \log C_3$$

$$\Rightarrow \quad Y = C_3 e^{(k-2)y} \qquad \ldots(4)$$

Hence from (1),

$$u(x, y) = (C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x})\, C_3 e^{(k-2)y} \qquad \ldots(5)$$

Applying the condition $u(0, y) = 0$ in (5), we get

$$u(0, y) = 0 = (C_1 + C_2)\, C_3 e^{(k-2)y}$$

$$\Rightarrow \quad C_1 + C_2 = 0 \quad \Rightarrow \quad C_2 = -C_1 \qquad \ldots(6)$$

($\because$ $C_3 = 0$ leads to trivial solution)

From (5), most general solution is

$$u(x, y) = \Sigma\, C_1 C_3 (e^{\sqrt{k}x} - e^{-\sqrt{k}x})\, e^{(k-2)y} \qquad \ldots(7)$$

$$\frac{\partial u}{\partial x} = \Sigma\, C_1 C_3 \sqrt{k} (e^{\sqrt{k}x} + e^{-\sqrt{k}x})\, e^{(k-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 1 + e^{-3y} = \Sigma C_1 C_3 \sqrt{k}(2)\, e^{(k-2)y} = \sum_{n=1}^{\infty} b_n\, e^{(k-2)y}$$

Comparing the coefficients, we get

(*i*) $b_1 = 1, \quad k - 2 = 0$

$$2C_1 C_3 \sqrt{k} = 1, \quad k = 2$$

$$\therefore \quad C_1 C_3 = \frac{1}{2\sqrt{2}}$$

(*ii*) $b_3 = -1, \quad k - 2 = -3$

$$2C_1 C_3 \sqrt{k} = 1, \quad k = -1$$

$$\therefore \quad C_1 C_3 = \frac{1}{2i}$$

Hence from (7), the particular solution is

$$u(x, y) = \frac{1}{2\sqrt{2}}\, (e^{\sqrt{2}x} - e^{-\sqrt{2}x}) + \frac{1}{2i} (e^{ix} - e^{-ix})\, e^{-3y}$$

$$\Rightarrow \quad u(x, y) = \frac{1}{\sqrt{2}} \sinh \sqrt{2}\, x + e^{-3y} \sin x.$$

EXERCISE A

NOTES

Solve by the method of separation of variables (1–10):

1. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0;\ u(x, 0) = 4e^{-x}$

2. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y};\ u(0, y) = 8e^{-3y}$

3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u;\ u(0, y) = 4e^{-y} - e^{-5y}$

4. $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - 2u;\ u(x, 0) = 10e^{-x} - 6e^{-4x}$

5. (*i*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (*ii*) $y^3\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = 0$

6. (*i*) $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u;\quad u(x, 0) = 6e^{-5x}$ (*ii*) $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$ given $u(0, y) = 3e^{-y} - e^{-5y}$

7. (*i*) $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$ (*ii*) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = 0.$

8. (*i*) $x\dfrac{\partial^2 u}{\partial x \partial y} + 2yu = 0$ (*ii*) $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u;\ u(x, 0) = 3e^{-5x} - 2e^{-3x}$

9. (*i*) $2\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} + 5u = 0;\quad u(0, y) = 2e^{-y}$ (*ii*) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t};\quad u(x, 0) = x^2(x^2 + 1)$

10. $\dfrac{\partial z}{\partial x} + \dfrac{\partial^2 z}{\partial y^2} = 0;\quad z(x, 0) = 0,\ z(x, \pi) = 0,\ z(0, y) = 4\sin 3y.$

11. Solve $\dfrac{\partial^2 u}{\partial x^2} = 2u + \dfrac{\partial u}{\partial y}$ using method of separation of variables subject to the conditions $u = 0$ and $\dfrac{\partial u}{\partial x} = e^{-3y}$ when $x = 0$ for all values of y.

12. Using method of separation of variables, obtain the solution of the equation $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ for which $\dfrac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ and $z = 0$ when y is an odd multiple of $\dfrac{\pi}{2}$.

Answers

1. $u(x, y) = 4e^{-x + \frac{3}{2}y}$

2. $u(x, y) = 8e^{-12x - 3y}$

3. $u(x, y) = 4e^{x - y} - e^{2x - 5y}$

4. $u(x, y) = 10e^{-(x + 3t)} - 6e^{-2(2x + 3t)}$

5. (*i*) $u(x, y) = c\left(\dfrac{x}{y}\right)^k$ (*ii*) $u(x, y) = ce^{p^2\left(\frac{y^4}{4} - \frac{x^3}{3}\right)}$

6. (*i*) $u(x, t) = 6e^{-5x - 3t}$ (*ii*) $u(x, y) = 3e^{x - y} - e^{2x - 5y}$

7. (*i*) $z(x, y) = cx^{3k}y^{2k}$ (*ii*) $u(x, y) = e^{-p^2 y}(A\cos px + B\sin px)$

8. (*i*) $u(x, y) = cx^k e^{-\left(\frac{y^2}{k}\right)}$ (*ii*) $u(x, y) = 3e^{-(5x + 3y)} - 2e^{-(3x + 2y)}$

9. (*i*) $u(x, y) = 2e^{-(x + y)}$ (*ii*) $u(x, t) = x^2(x^2 + 1)e^{kt}$

10. $z(x, y) = 4e^{9x}\sin 3y$

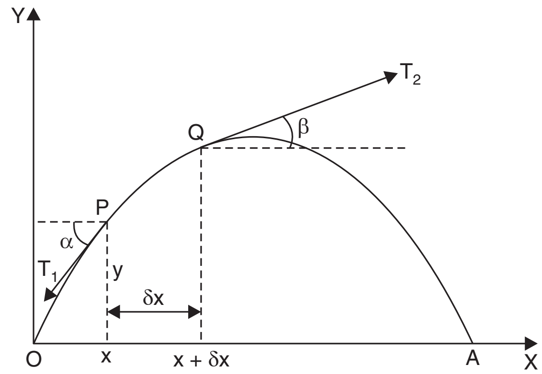
11. $u(x, y) = e^{-3y}\sin x$

12. $z(x, y) = \cos y(1 + \cos x)$

NOTES

VIBRATIONS OF A STRETCHED STRING, ONE DIMENSIONAL WAVE EQUATION $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Consider a uniform elastic string of length l stretched tightly between two points O and A and displaced slightly from its equilibrium position OA. Taking the end O as the origin, OA as the x-axis and a perpendicular line through O as the y-axis, we shall find the displacement y as a function of the distance x and the time t.



We shall obtain the equation of motion for the string under the following assumptions:

(*i*) The motion takes place entirely in the xy-plane and each particle of the string moves perpendicular to the equilibrium position OA of the string.

(*ii*) The string is perfectly flexible and does not offer resistance to bending.

(*iii*) The tension in the string is so large that the forces due to weight of the string can be neglected.

(*iv*) The displacement y and the slope $\frac{\partial y}{\partial x}$ are small, so that their higher powers can be neglected.

Let m be the mass per unit length of the string. Consider the motion of an element PQ of length δs. Since the string does not offer resistance to bending (by assumption), the tensions T_1 and T_2 at P and Q respectively are tangential to the curve.

Since, there is no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (constant)} \qquad ...(1)$$

Mass of element PQ is $m\delta s$. By Newton's second law of motion, the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin \beta - T_1 \sin \alpha$$

or $$\frac{m\delta s}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} \qquad \text{[By using (1)]}$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta s} (\tan \beta - \tan \alpha)$$

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m\delta x} \left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right]$$

[Since $\delta s = \delta x$ to a first approximation and $\tan \alpha$ and $\tan \beta$ are the slopes of the curve of the string at x and $x + \delta x$]

NOTES

or $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right] = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}, \text{ as } \delta x \to 0$$

or $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \qquad \text{where } c^2 = \frac{T}{m}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the **one dimensional wave equation.**

The *boundary conditions* which the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ has to satisfy are :

$$\left.\begin{array}{l}(i)\ y = 0 \text{ when } x = 0\\ (ii)\ y = 0 \text{ when } x = l\end{array}\right]$$. These should be satisfied for every value of t.

If the string is made to vibrate by pulling it into a curve $y = f(x)$ and then releasing it, the *initial conditions* are

(*i*) $y = f(x)$ when $t = 0$ (*ii*) $\frac{\partial y}{\partial t} = 0$ when $t = 0$.

SOLUTION OF THE WAVE EQUATION

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let $$y = XT \qquad ...(2)$$

where X is a function of x only and T is a function of t only, be a solution of (1)

Then, $$\frac{\partial^2 y}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in (1), we have $XT'' = c^2X''T$

Separating the variables, we get $$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} \qquad ...(3)$$

Now the LHS of (3) is a function of x only and the RHS is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary linear differential equations.

$$X''k - X = 0 \quad \text{and} \quad T'' - kc^2T = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3e^{cpt} + c_4e^{-cpt}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \quad T = c_3 \cos cpt + c_4 \sin cpt$$

(*iii*) *When k = 0*

$$X = c_1x + c_2 \qquad T = c_3t + c_4$$

Thus the various possible solutions of the wave equation (1) are:

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt})$$

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$
$$y = (c_1 x + c_2)(c_3 t + c_4)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t. Therefore, the solution must involve trigonometric terms.

Accordingly $y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ ...(5)

is the only suitable solution of the wave equation and it corresponds to $k = -p^2$.

Now, we apply the boundary conditions. Since the string is fastened at the ends $x = 0$ and $x = l$, we have two boundary conditions

$$y(0, t) = 0 \quad \text{and} \quad y(l, t) = 0 \quad \text{for all } t \geq 0.$$

Using $y = 0$ when $x = 0$ and $y = 0$ when $x = l$,

we get $0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$ ...(6)

and $0 = (c_1 \cos pl + c_2 \sin pl)(c_3 \cos cpt + c_4 \sin cpt)$ ...(7)

From (6), we have $c_1 = 0$ and equation (7) reduces to

$$c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$$

which is satisfied when $\sin pl = 0$ or $pl = n\pi$ or $p = \frac{n\pi}{l}$, where $n = 1, 2, 3, ...$

$\therefore$ A solution of the wave equation satisfying the boundary conditions is

$$y = c_2 \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

or $$y_n = \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \text{ where } n = 1, 2, 3, ...$$

on replacing c_2c_3 by a_n and c_2c_4 by b_n.

These functions are called the **eigen functions or characteristic functions.**

The values $\lambda_n = \frac{nc\pi}{l}$ are called the **eigen values or characteristic values** of the vibrating string. The set $\{\lambda_1, \lambda_2, \lambda_3, ...\}$ is called the **spectrum.**

Adding up the solutions for different values of n, we get

$$y = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad ...(8)$$

which is the general solution.

Now we apply the initial conditions the form of motion of the string depends on its initial deflection (*i.e.*, y at time $t = 0$) and its initial velocity $\left(i.e., \frac{\partial y}{\partial t} \text{ at } t = 0 \right)$.

Thus we have two initial conditions

$y(x, 0) = f(x)$ and $y_t(x, 0) = 0$, $(0 \leq x \leq l)$

Using $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$, when $t = 0$,

we have $$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad ...(9)$$

and $$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l} \quad ...(10)$$

Since equation (9) represents half range Fourier sine series for $f(x)$ in the interval $(0, l)$ we have

$$a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad ...(11)$$

From (10), $b_n = 0$, for all n.

Hence (8) reduces to $y = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$ ...(12)

where a_n is given by (11) when $f(x)$ *i.e.*, $y(x, 0)$ is known.

Note: $\sin pl = 0 \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$ where n is an integer.

For negative values of n we obtain the same solutions as for corresponding positive values of n except for a minus sign, because $\sin(-\alpha) = -\sin\alpha$

This is the reason for taking $n = 1, 2, 3, \ldots$.

TRANSFORMING NON-HOMOGENEOUS BCs TO HOMOGENEOUS ONES

The method of separation of variable discussed earlier is very powerful but not applicable to all problems. It is applicable only to problems with **zero boundary conditions (BCs) called homogeneous BCs.** Here we shall learn transformation formulae to convert two different types of non-homogeneous BC*s* to homogeneous BCs.

I. Dirichlet Boundary Conditions (u prescribed on the boundary curve of region R)

In this first type of boundary conditions, the displacements $u(0, t) = \alpha$ and $u(l, t) = \beta$ of a vibrating string of length l are given. Consider the initial boundary value problem (IBVP):

PDE	$u_{tt} = c^2 u_{xx}$,	$0 < x < l$,	$t > 0$
BCs	$u(0, t) = \alpha$	$u(l, t) = \beta$,	$t > 0$ (Non-homogeneous BCs)
ICs	$u(x, 0) = f(x)$,	$u_t(x, 0) = g(x)$	

We cannot solve this problem by the method of separation of variables since the BCs are non-homogeneous.

To convert the above non-homogeneous BCs to homogeneous BCs, we use the conversion formula

$$u(x, t) = \left[\alpha + \left(\frac{\beta - \alpha}{l}\right)x\right] + v(x, t) \qquad ...(1)$$

in the original problem and arrive at a new problem in $v(x, t)$.

Clearly, from (1), we have

$$u_t = v_t \quad , \qquad u_{tt} = v_{tt}$$

$$u_x = \frac{\beta - \alpha}{l} + v_x \quad , \qquad u_{xx} = v_{xx}$$

so that $u_{tt} = c^2 u_{xx}$ transforms into $v_{tt} = c^2 v_{xx}$

Also, $u(0, t) = \alpha \Rightarrow \alpha = \alpha + v(0, t) \Rightarrow v(0, t) = 0$

$$u(l, t) = \beta \Rightarrow \beta = \alpha + \left(\frac{\beta - \alpha}{l}\right) l + v\,(l, t)$$

$$\Rightarrow v(l, t) = 0$$

And $u(x, 0) = f(x) \Rightarrow f(x) = \left[\alpha + \left(\frac{\beta - \alpha}{l}\right)x\right] + v(x, 0)$

$$\Rightarrow v(x, 0) = f(x) - \left[\alpha + \left(\frac{\beta - \alpha}{l}\right)x\right]$$

$$u_t\,(x, 0) = g(x) \Rightarrow v_t\,(x, 0) = g(x)$$

The new PDE with homogeneous BCs is

PDE $v_{tt} = c^2 v_{xx}$, $0 < x < l$, $t > 0$

BCs $v(0, t) = 0$, $v\,(l, t) = 0$, $t > 0$

ICs $v(x, 0) = f(x) - \left[\alpha + \left(\frac{\beta - \alpha}{l}\right)x\right]$ (Homogeneous BCs) $v_t\,(x, 0) = g(x)$

Now using method of separation of variables, $v(x, t)$ can be found out. Subsequently, $u(x, t)$ follows from (1).

II. Neumarnn Boundary Conditions (u_x prescribed on C)

In this second type of boundary conditions, $u_x\,(0, t) = \alpha$ and $u_x(l, t) = \beta$ are given. Consider the initial boundary value problem (IBVP):

PDE $u_{tt} = c^2\,u_{xx}$, $0 < x < l$, $t > 0$

BCs $u_x\,(0, t) = \alpha$, $u_x\,(l, t) = \beta$, $t > 0$

ICs $u(x, 0) = f(x)$, $u_t\,(x, 0) = g(x)$

To convert the above non-homogeneous BCs to homogeneous BCs, we use the conversion formula

$$u(x, t) = \alpha x + \left(\frac{\beta - \alpha}{2l}\right)x^2 + c^2\left(\frac{\beta - \alpha}{2l}\right)t^2 + v(x, t) \qquad ...(1)$$

in the original problem and arrive at a new problem in $v(x, t)$.

Clearly, from (1), we have

$$u_t = c^2\left(\frac{\beta - \alpha}{l}\right)t + v_t, \qquad u_{tt} = c^2\left(\frac{\beta - \alpha}{l}\right) + v_{tt}$$

$$u_x = \alpha + \left(\frac{\beta - \alpha}{l}\right)x + v_x, \qquad u_{xx} = \left(\frac{\beta - \alpha}{l}\right) + v_{xx}$$

$$\Rightarrow \quad c^2\,u_{xx} = c^2\left(\frac{\beta - \alpha}{l}\right) + c^2\,v_{xx}$$

so that $u_{tt} = c^2\,u_{xx}$ transforms into $v_{tt} = c^2\,v_{xx}$

Also, $u_x\,(0, t) = \alpha \Rightarrow \alpha = \alpha + v_x\,(0, t)$

$\Rightarrow v_x\,(0, t) = 0$

$$u_x\,(l, t) = \beta \Rightarrow \beta = \alpha + \left(\frac{\beta - \alpha}{l}\right) l + v_x\,(l, t)$$

$\Rightarrow v_x(l, t) = 0$

NOTES

And $\quad u(x, 0) = f(x) \quad \Rightarrow \quad f(x) = \alpha x + \left(\frac{\beta - \alpha}{2l}\right)x^2 + v(x, 0)$

$$\Rightarrow \quad v(x, 0) = f(x) - \left[\alpha x + \left(\frac{\beta - \alpha}{2l}\right)x^2\right]$$

$$u_t(x, 0) = g(x) \quad \Rightarrow \quad g(x) = v_t(x, 0)$$

The new PDE with homogeneous BCs is

PDE $\quad v_{tt} = c^2 v_{xx}, \quad 0 < x < l, \quad t > 0$

BCs $\quad v_x(0, t) = 0, \quad v_x(l, t) = 0, \quad t > 0$

ICs $\quad v(x, 0) = f(x) - \left[\alpha x + \left(\frac{\beta - \alpha}{2l}\right)x^2\right], \quad v_t(x, 0) = g(x)$

Now using method of separation of variables, $v(x, t)$ can be found out. Subsequently, $u(x, t)$ follows from (1).

D'ALEMBERT'S SOLUTION OF THE WAVE EQUATION

The solution of the wave equation

$$u_{tt} = c^2 u_{xx} \qquad ...(1)$$

can be easily obtained by introducing two new variables

$$v = x + ct \quad \text{and} \quad w = x - ct \qquad ...(2)$$

so that u becomes a function of v and w.

Now, $\quad u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} = \frac{\partial u}{\partial v}(c) + \frac{\partial u}{\partial w}(-c)$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = c\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right) \quad \Rightarrow \quad \frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)$$

$$\therefore \quad u_{tt} = \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} - \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}\right)$$

$$\Rightarrow \quad u_{tt} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right) \qquad ...(3)$$

Also, $\quad u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}(1) + \frac{\partial u}{\partial w}(1)$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} \quad \Rightarrow \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$$

$$\therefore \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w \partial v} + \frac{\partial^2 u}{\partial w^2}$$

$$\Rightarrow \quad u_{xx} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \qquad ...(4)$$

Using (3) and (4), (1) reduces to

$$c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right) = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right)$$

or $$4\frac{\partial^2 u}{\partial v \partial w} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial w \partial v} = 0$$

$\Rightarrow$ $$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right) = 0$$

Integrating w.r.t. w, (v constant)

$$\frac{\partial u}{\partial v} = h(v)$$

where $h(v)$ is an arbitrary function of v.

Integrating w.r.t. v, (w constant)

$$u = \int h(v)\, dv + \psi(w)$$

$$= \phi(v) + \psi(w)$$

where $\psi(w)$ is an arbitrary function of w.

or $$u(x, t) = \phi(x + ct) + \psi(x - ct)$$

This is known as D'Alembert's solution of the wave equation.

D'ALEMBERT's SOLUTION SATISFYING INITIAL CONDITIONS

The physical model that controls the wave motion of a very long string is governed by a PDF and initial conditions only, no BCs in the absence of boundaries. The method of separation of variables is not applicable in this case.

However, D'Alembert's solution allows us to solve the initial value problem on an infinite domain.

Now, we solve the wave equation

$$u_{tt} = c^2\, u_{xx},\ -\infty < x < \infty,\ t > 0 \qquad \text{...(1)}$$

given that initial deflection $u(x, 0) = f(x)$ and initial velocity $u_t(x, 0) = g(x)$.

D' Alembert's solution of (1) is

$$u(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \text{...(2)}$$

Using the initial condition $u(x, 0) = f(x)$ in (2), we get

$$f(x) = \phi(x) + \psi(x) \qquad \text{...(3)}$$

From (2), $$u_t(x, t) = c\phi'(x + ct) - c\psi'(x - ct) \qquad \text{...(4)}$$

Using the initial condition $u_t(x, 0) = g(x)$ in (4), we get

$$g(x) = c\phi'(x) - c\psi'(x)$$

or $$\frac{1}{c}\, g(x) = \phi'(x) - \psi'(x)$$

Integrating both sides w.r.t. x, we get

$$\frac{1}{c}\int_a^x g(\theta)\, d\theta = \phi(x) - \psi(x) \qquad \text{...(5)}$$

where a is arbitrary.

NOTES

[We have used Fundamental theorem of calculus, *viz.*

$$\frac{d}{dt}\left[\int_a^x F(t)\,dt = F(x)\right]$$

Solving (3) and (5),

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(\theta)\,d\theta \quad ...(6)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(\theta)\,d\theta \quad ...(7)$$

Replacing x by $x + ct$ in (6) and $x - ct$ in (7), we have

$$\phi(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_a^{x+ct} g(\theta)\,d\theta \quad ...(8)$$

$$\psi(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c}\int_a^{x-ct} g(\theta)\,d\theta$$

$$= \frac{1}{2} f(x - ct) + \frac{1}{2c}\int_{x-ct}^{a} g(\theta)\,d\theta \quad ...(9)$$

Adding (8) and (9),

$$\phi(x + ct) + \psi(x - ct) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\left[\int_{x-ct}^{a} g(\theta)\,d\theta + \int_a^{x+ct} g(\theta)\,d\theta\right]$$

$$= \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\theta)\,d\theta$$

Using (2), we get

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\theta)\,d\theta$$

which is the required solution satisfying initial conditions.

Particular Case: If the string is initially at rest, $u_t(x, 0) = 0$ so that $g(x) = 0$ and the solution reduces to

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

For example. (1) Let us find D'Alembert's solution of

PDE: $u_{tt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$

IC: $u(x, 0) = x^2$, $u_t(x, 0) = \sin x$

Here $c^2 = 1 \Rightarrow c = 1$

$f(x) = x^2$, $g(x) = \sin x$

∴ D' Alembert's solution is

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - t)] + \frac{1}{2c}\int_{x-t}^{x+t} g(\theta)\,d\theta$$

$$= \frac{1}{2}[f(x + t)^2 + f(x - t)^2] + \frac{1}{2}\int_{x-t}^{x+t} \sin\theta\,d\theta$$

$$= x^2 + t^2 - \frac{1}{2}[\cos(x + t) - \cos(x - t)]$$

$$= x^2 + t^2 - \frac{1}{2}(-2\sin x \sin t)$$

or $$u(x, t) = x^2 + t^2 + \sin x \sin t.$$

(2) Let us find D' Alembert's solution of

PDE: $u_{tt} = 4c_{xx}$, $-\infty < x < \infty$, $t > 0$

IC: $u(x, 0) = x^4$, $u_t(x, 0) = 0$

Here, $c^2 = 4 \Rightarrow c = 2$

$f(x) = x^4$, $g(x) = 0$

$\therefore$ D'Alembert's solution is

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

or

$$u(x, t) = \frac{1}{2}[(x + 2t)^4 + (x - 2t)^4]$$

DUHAMEL'S PRINCIPLE FOR ONE DIMENSIONAL WAVE EQUATION

In D'Alembert's solution of the wave equation satisfying initial conditions

(*i*) **PDE is homogeneous**

(*ii*) at least one of $f(x)$ and $g(x)$ is non-zero.

Duhamel's principle gives the unique solution of the *non-homogeneous wave equation when both f(x) and g(x) are zero.*

Statement. The unique solution of the non-homogeneous wave equation

$$u_{tt} = c^2 u_{xx} + h(x, t), \quad -\infty < x < \infty,\ t > 0$$

with the initial deflection $u(x, 0) = 0$ and initial velocity $u_t(x, 0) = 0$ is given by

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{x - c(t-s)}^{x + c(t-s)} h(r, s)\, dr ds.$$

For example. Let us find the solution of

PDE: $u_{tt} = u_{xx} + x - t$, $-\infty < x < \infty$, $t > 0$

IC: $u(x, 0) = 0$, $u_t(x, 0) = 0$

Here $c^2 = 1 \Rightarrow c = 1$ and $h(x, t) = x - t$

By Duhamel's principle, the unique solution is

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{x - c(t-s)}^{x + c(t-s)} h(r, s)\, dr ds$$

$$= \frac{1}{2}\int_0^t \int_{x - (t-s)}^{x + (t-s)} (r - s)\, dr ds$$

$$= \frac{1}{2}\int_0^t \left[\frac{r^2}{2} - sr\right]_{x - (t-s)}^{x + (t-s)} ds$$

$$= \frac{1}{2}\int_0^t \left(\frac{[x + (t-s)]^2 - [x - (t-s)]^2}{2} - s[x + (t-s) - x + (t-s)]\right) ds$$

$$= \frac{1}{2}\int_0^t [2x(t - s) - 2s(t - s)]\, ds$$

$$= \frac{1}{2}\int_0^t [2s^2 - 2(t + x)s + 2tx]\, ds$$

NOTES

$$= \left[\frac{s^3}{3} - (t+x)\frac{s^2}{2} + txs\right]_0^t$$

$$= \frac{t^3}{3} - (t+x)\frac{t^2}{2} + t^2x$$

$$= -\frac{t^3}{6} + \frac{t^2x}{2}$$

$\Rightarrow \quad u(x, t) = -\dfrac{t^3}{6} + \dfrac{t^2x}{2}$ is the required unique solution.

SOLVED EXAMPLES

Example 9. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form* $y = A \sin \dfrac{\pi x}{l}$ *from which it is released at time t = 0. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x, t) = A \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}.$$

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

Since, the string is stretched between two fixed points (0, 0) and (l, 0) hence the displacement of the string at these points will be zero

$$\therefore \quad y(0, t) = 0 \quad \text{...(2)}$$

and
$$y(l, t) = 0 \quad \text{...(3)}$$

Since, the string is released from rest hence its initial velocity will be zero

$$\therefore \quad \frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0 \quad \text{...(4)}$$

Since, the string is displaced from its initial position at time $t = 0$ hence the initial displacement is

$$y(x, 0) = A \sin \frac{\pi x}{l} \quad \text{...(5)}$$

Conditions (2), (3), are the boundary conditions and (4), (5) are initial conditions.

Let us now proceed to solve equation (1),

Let $\quad y = XT. \quad$ ...(6)

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(XT) = X\frac{dT}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(X\frac{dT}{dt}\right) = X\frac{d^2T}{dt^2}.$$

NOTES

Similarly, $\frac{\partial^2 y}{\partial x^2} = \text{T}\,\frac{d^2\text{X}}{dx^2}.$

Substituting the above in equation (1), we get

$$\text{X}\frac{d^2\text{T}}{dt^2} = c^2\text{T}\,\frac{d^2\text{X}}{dx^2} \quad \Rightarrow \quad \text{XT}'' = c^2\text{TX}''$$

Case I. $\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} = -\,p^2$ (say)

(*i*) $\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = -\,p^2$

$$\frac{d^2\text{T}}{dt^2} + c^2p^2\text{T} = 0.$$

Auxiliary equation is $m^2 + c^2p^2 = 0$

$$m^2 = c^2p^2i^2$$

$$m = \pm\, cpi$$

∴ C.F. $= c_1 \cos cpt + c_2 \sin cpt$

P.I. = 0

∴ $\text{T} = \text{C.F.} + \text{P.I.} = c_1 \cos cpt + c_2 \sin cpt$ ...(7)

(*ii*) $\frac{\text{X}''}{\text{X}} = -\,p^2 \quad \Rightarrow \quad \frac{d^2\text{X}}{dx^2} + p^2\text{X} = 0.$

Auxiliary equation is $m^2 + p^2 = 0$

$$m = \pm\, pi$$

C.F. $= c_3 \cos px + c_4 \sin px$

P.I. = 0

∴ $\text{X} = c_3 \cos px + c_4 \sin px$ ...(8)

Hence, $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(9)

Case II. $\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} = p^2$ (say)

(*i*) $\frac{1}{c^2}\frac{\text{T}''}{\text{T}} = p^2 \quad \Rightarrow \quad \frac{d^2\text{T}}{dt^2} - p^2c^2\text{T} = 0$

Auxiliary equation is $m^2 - p^2c^2 = 0 \quad \Rightarrow \quad m = \pm\, pc$

∴ C.F. $= c_5\, e^{pct} + c_6\, e^{-pct}$

P.I. = 0

∴ $\text{T} = c_5\, e^{pct} + c_6\, e^{-pct}.$

(*ii*) $\frac{\text{X}''}{\text{X}} = p^2 \quad \Rightarrow \quad \frac{d^2\text{X}}{dx^2} - p^2\text{X} = 0$

Auxiliary equation is

$$m^2 - p^2 = 0 \quad \Rightarrow \quad m = \pm\, p$$

∴ C.F. $= c_7\, e^{px} + c_8\, e^{-px}$

P.I. = 0

∴ $\text{X} = c_7\, e^{px} + c_8\, e^{-px}$

Hence, $y(x, t) = (c_5\, e^{pct} + c_6\, e^{-pct})(c_7\, e^{px} + c_8\, e^{-px})$ ...(10)

NOTES

Case III. $$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = 0 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow T'' = 0 \quad \text{or} \quad \frac{d^2T}{dt^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_9 + c_{10}t$$

$$\text{P.I.} = 0$$

$\therefore$ $$T = c_9 + c_{10}t$$

(*ii*) $$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \quad \text{or} \quad \frac{d^2X}{dx^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_{11} + c_{12}x$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_{11} + c_{12}x$$

Hence, $$y(x, t) = (c_9 + c_{10}t)(c_{11} + c_{12}x) \quad \text{...(11)}$$

Out of these three above solutions (9), (10) and (11), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, the solution must contain periodic functions. Hence the solution which contains trigonometric terms must be the required solution.

Hence solution (9) is the general solution of one dimensional wave equation given by equation (1).

Now, $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$$

Applying the boundary condition,

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $$c_3 = 0.$$

$\therefore$ From (9), $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad \text{...(12)}$$

Again, $$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \quad (n \in I)$$

$\therefore$ $$p = \frac{n\pi}{l}.$$

Hence from (12), $$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \text{...(13)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $$c_2 = 0,$$

$\therefore$ From (13), $$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{...(14)}$$

$$y(x, 0) = A \sin \frac{\pi x}{l} = c_1 c_4 \sin \frac{n\pi x}{l}$$

NOTES

$\Rightarrow \qquad c_1c_4 = \text{A}, n = 1.$ | Comparing

Hence from (14), $y(x, t) = \text{A} \cos \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l}$

which is the required solution.

Example 10. *Show how the wave equation* $c^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$ *can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between x = 0 and x = l are given by* $y = f(x)$ *and* $\dfrac{\partial y}{\partial t} = g(x)$, *determine the constants in the series solution.*

Sol. The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(1)

Let $\qquad y = \text{XT}$ ...(2)

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(\text{XT}) = \text{X}\frac{d\text{T}}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \text{X}\frac{d^2\text{T}}{dt^2}$$

Similarly, $\qquad \dfrac{\partial^2 y}{\partial x^2} = \text{T}\dfrac{d^2\text{X}}{dx^2}.$

Substituting in (1), we get

$$\text{X}\frac{d^2\text{T}}{dt^2} = c^2\text{T}\frac{d^2\text{X}}{dx^2} \quad \Rightarrow \quad \text{XT}'' = c^2\text{TX}''$$

$$\Rightarrow \qquad \frac{1}{c^2}\frac{\text{T}''}{\text{T}} = \frac{\text{X}''}{\text{X}} \qquad \text{...(3)}$$

Case I. When $\dfrac{1}{c^2}\dfrac{\text{T}''}{\text{T}} = \dfrac{\text{X}''}{\text{X}} = p^2$ (say)

(*i*) $\qquad \dfrac{1}{c^2}\dfrac{\text{T}''}{\text{T}} = p^2 \quad \Rightarrow \quad \dfrac{d^2\text{T}}{dt^2} - p^2c^2\text{T} = 0.$

Auxiliary equation is

$$m^2 - p^2c^2 = 0$$

$$m = \pm pc$$

$$\text{C.F.} = c_1 e^{pct} + c_2 e^{-pct}$$

$$\text{P.I.} = 0$$

$$\therefore \qquad \text{T} = \text{C.F.} + \text{P.I.} = c_1 e^{pct} + c_2 e^{-pct}$$

(*ii*) $\qquad \dfrac{\text{X}''}{\text{X}} = p^2 \quad \Rightarrow \quad \dfrac{d^2\text{X}}{dx^2} - p^2\text{X} = 0.$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p$$

$$\text{C.F.} = c_3 e^{px} + c_4 e^{-px}$$

$$\text{P.I.} = 0.$$

$$\therefore \qquad \text{X} = \text{C.F.} + \text{P.I.} = c_3 e^{px} + c_4 e^{-px}.$$

Hence, the solution is

$$y = XT = (c_1 e^{pct} + c_2 e^{-pct})(c_3 e^{px} + c_4 e^{-px}). \quad ...(4)$$

Case II. When

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = -p^2 \Rightarrow \frac{d^2T}{dt^2} + p^2c^2T = 0.$$

Auxiliary equation is

$$m^2 + p^2c^2 = 0 \Rightarrow m = \pm pci$$

$\therefore$ $$\text{C.F.} = (c_5 \cos pct + c_6 \sin cpt)$$

$$\text{P.I.} = 0.$$

$\therefore$ $$T = \text{C.F.} + \text{P.I.} = c_5 \cos cpt + c_6 \sin cpt$$

(*ii*) $$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0.$$

Auxiliary equation is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_7 \cos px + c_8 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_7 \cos px + c_8 \sin px.$$

Hence, the solution is

$$y = XT = (c_5 \cos cpt + c_6 \sin cpt)(c_7 \cos px + c_8 \sin px) \quad ...(5)$$

Case III. When, $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = 0$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow \frac{d^2T}{dt^2} = 0$$

$\Rightarrow$ $$T = c_9 + c_{10}t$$

(*ii*) $$\frac{X''}{X} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$

$\Rightarrow$ $$X = c_{11} + c_{12}\, x.$$

Hence, the solution is

$$y(x, t) = (c_9 + c_{10}t)(c_{11} + c_{12}x) \quad ...(6)$$

Of the above three solutions given by (4), (5) and (6), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t therefore the solution must involve trigonometric terms hence solution (5) is the required solution.

Boundary conditions are

$$y(0, t) = 0, \qquad y(l, t) = 0$$

Initial conditions are $y = f(x)$

and $$\frac{\partial y}{\partial t} = g(x)$$

From equation (5), $$y(0, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$$

$$0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$$

$\Rightarrow$ $$c_7 = 0.$$

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Hence from (5), $y(x, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin px$ ...(7)

$$y(l, t) = 0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \; (n \in \text{I}) \;\Rightarrow\; p = \dfrac{n\pi}{l}$.

$\therefore$ From (7), $$y(x, t) = \left(c_5 \cos \frac{n\pi ct}{l} + c_6 \sin \frac{n\pi ct}{l}\right) c_8 \sin \frac{n\pi x}{l} \quad ...(8)$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

where $c_5 c_8 = a_n$ and $c_6 c_8 = b_n$

The general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \quad ...(9)$$

$$y(x, 0) = f(x) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

where $$a_n = \frac{2}{l} \int_0^l f(x) \,.\, \sin \frac{n\pi x}{l}\, dx \quad ...(10)$$

From (9), $$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{1}^{\infty} \left(-n\, a_n \sin \frac{n\pi ct}{l} + n\, b_n \cos \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x) = \frac{\pi c}{l} \sum_{1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

where $$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) \,.\, \sin \frac{n\pi x}{l}\, dx$$

$\Rightarrow$ $$b_n = \frac{2}{n\pi c} \int_0^l g(x) \,.\, \sin \frac{n\pi x}{l}\, dx. \quad ...(11)$$

Hence, the required solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

where $$a_n = \frac{2}{l} \int_0^l f(x) \,.\, \sin \frac{n\pi x}{l}\, dx$$

and $$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx.$$

Example 11. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \dfrac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.*

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2)$$

| Refer Sol. of Ex. 1

NOTES

Boundary conditions are

$$y(0, t) = 0, \quad \text{...(3)}$$

$$y(l, t) = 0 \quad \text{...(4)}$$

Initial conditions are

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0, \quad \text{...(5)}$$

and $$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} \quad \text{...(6)}$$

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $$c_3 = 0$$

$\therefore$ From (2), $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad \text{...(7)}$$

Again, $$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \quad (n \in \text{I})$$

$\therefore$ $$p = \frac{n\pi}{l}$$

Hence, from (7),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \text{...(8)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[- c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$c_2 = 0.$$

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{...(9)}$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4}\right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing, we get

$$b_1 = \frac{3y_0}{4},\ b_2 = 0,\ b_3 = \frac{-y_0}{4},\ b_4 = b_5 = \ldots = 0$$

Hence, from (9),

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

NOTES

Example 12. *A tightly stretched flexible string has its ends fixed at x = 0 and x = l. At time t = 0, the string is given a shape defined by F(x) = μx(l – x), μ is a constant and then released. Find the displacement y(x, t) of any point x of the string at any time t > 0.*

Sol. The wave equation is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(1)

The solution of equation (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2)

(Refer Sol. of Ex. 1)

Boundary conditions are $y(0, t) = 0$ ...(3)

$y(l, t) = 0$ ..(4)

Initial conditions are

$$\frac{\partial y}{\partial t} = 0 \quad ...(5)$$

and $y(x, 0) = \mu x(l - x)$ ...(6)

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \mathrm{I})$

$$p = \frac{n\pi}{l}.$$

From (7), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ ...(8)

Now from (7), $\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] . c_4 \sin \frac{n\pi x}{l}$

At $t = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (8), $y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

$\Rightarrow$ $y(x, t) = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ where $c_1 c_4 = b_n$.

The most general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(9)$$

$$y(x, 0) = \mu(lx - x^2) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^l \mu\,(lx - x^2) \sin \frac{n\pi x}{l}\, dx$

$$= \frac{2\mu}{l}\left[\left\{(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l (l - 2x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

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$$= \frac{2\mu}{l}\left[\frac{l}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2\mu}{n\pi}\left[\left\{(l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l (-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right]$$

$$= \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4\mu l}{n^2\pi^2}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l = \frac{4\mu l^2}{n^3\pi^3}(-\cos n\pi + 1) = \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n].$$

$\therefore$ From (9),
$$y(x, t) = \frac{4\mu l^2}{\pi^3}\sum_1^\infty \frac{[1-(-1)^n]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}.$$

Example 13. *A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by*

and
$$f(x) = \begin{cases}\left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2}\\ \left(\frac{2k}{l}\right)(l-x), & \frac{l}{2} < x < l\end{cases}$$

Find the deflection of the string at any time.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of eqn. (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are, $y(0, t) = 0$, $y(l, t) = 0$

Initial conditions are

$$\frac{\partial y}{\partial t} = 0,$$

and
$$y(x, 0) = \begin{cases}\frac{2k}{l}x, & 0 < x < \frac{l}{2}\\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l\end{cases}$$

From (2), $y(0, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(3)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi;\ n \in I$

$$p = \frac{n\pi}{l}.$$

$\therefore$ From (3), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ ...(4)

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (4), $y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

$= b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ (where $c_1 c_4 = b_n$) ...(5)

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(6)$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{[From (6)]}$$

where $b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2k}{l}(l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} - \int_0^{l/2} 1 . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right.$$

$$\left. + \left\{(l - x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l} (-1) . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l}{n\pi} . \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} + \frac{l}{2} . \frac{l}{n\pi} \cos \frac{n\pi}{2} - \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\sin n\pi - \sin \frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2} \sin \frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

$\therefore$ From (6), $y(x, t) = \frac{8k}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$

Example 14. *A tightly stretched violin string of length l and fixed at both ends is plucked at* $x = \frac{l}{3}$ *and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and any time t after the string is released from rest.*

Sol. One Dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) (Refer Sol. of Ex. 1)

Eqn. of line OC is $\quad y - 0 = \dfrac{a - 0}{\frac{l}{3} - 0}(x - 0)$

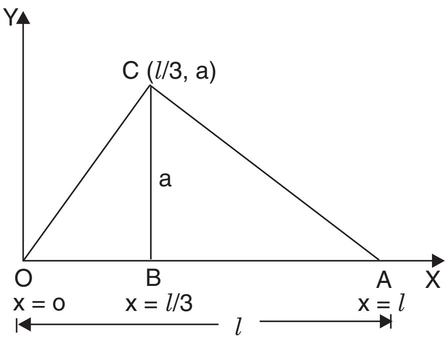
$$y = \frac{3a}{l} x \quad ...(3)$$

Eqn. of line CA is $\quad y - a = \dfrac{0 - a}{l - l/3}\left(x - \dfrac{l}{3}\right)$

$$y - a = \frac{-a}{\left(\frac{2l}{3}\right)}\left(x - \frac{l}{3}\right) = -\frac{3a}{2l}\left(x - \frac{l}{3}\right)$$

$$y - a = -\frac{3ax}{2l} + \frac{a}{2}$$

$$y = -\frac{3ax}{2l} + \frac{3a}{2} = \frac{3a}{2}\left(1 - \frac{x}{l}\right) \quad ...(4)$$



Hence the boundary conditions are

$$y(0, t) = 0 \quad ...(5)$$

$$y(l, t) = 0 \quad ...(6)$$

Initial conditions are

$$\frac{\partial y}{\partial t} = 0, \quad ...(7)$$

and

$$y(x, 0) = \begin{cases} \dfrac{3ax}{l}, & 0 < x < l/3 \\ \dfrac{3a}{2}\left(1 - \dfrac{x}{l}\right), & \dfrac{l}{3} < x < l \end{cases} \quad ...(8)$$

From (2), $\quad y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (2), $\quad y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad ...(9)$

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi (n \in I).$

$\Rightarrow \quad p = \dfrac{n\pi}{l}.$

$$\therefore \quad y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(10)$$

NOTES

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin\frac{n\pi ct}{l} + c_2 \cos\frac{n\pi ct}{l}\right] c_4 \sin\frac{n\pi x}{l}.$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin\frac{n\pi x}{l}\right]$$

$\Rightarrow$ $\qquad c_2 = 0.$

$\therefore$ $\qquad y(x, t) = c_1 c_4 \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} = b_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$

The most general solution is

$$y(x, t) = \sum_1^{\infty} b_n \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \qquad \ldots(11)$$

From (11), $\qquad y(x, 0) = \sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$, where

$$b_n = \frac{2}{l}\int_0^l y(x, 0)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/3}\frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{l}\frac{3a}{2}\left(1-\frac{x}{l}\right)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} x\sin\frac{n\pi x}{l}\,dx + \frac{3a}{2}\int_{l/3}^{l}\left(1-\frac{x}{l}\right)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left(x\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}\right)_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{3a}{l}\left[\left(\left(1-\frac{x}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right)_{l/3}^{l} - \int_{l/3}^{l}\left(-\frac{1}{l}\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l}{n\pi}.\frac{l}{3}\cos\frac{n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/3}\right]$$

$$+ \frac{3a}{l}\left[\frac{l}{n\pi}.\frac{2}{3}\cos\frac{n\pi}{3} - \frac{1}{n\pi}.\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right] + \frac{3a}{l}\left[\frac{2l}{3n\pi}\cos\frac{n\pi}{3} - \frac{l}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{6a}{n\pi}\left[\frac{-1}{3}\cos\frac{n\pi}{3} + \frac{1}{n\pi}\sin\frac{n\pi}{3}\right] + \frac{6a}{n\pi}\left[\frac{1}{3}\cos\frac{n\pi}{3}\right] + \frac{3a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

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$$\Rightarrow \qquad b_n = \frac{9a}{n^2\pi^2} \sin \frac{n\pi}{3}$$

$$\therefore \quad \text{From (11),} \quad y(x, t) = \frac{9a}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

Example 15. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

Sol. The equation for the vibration of the string is

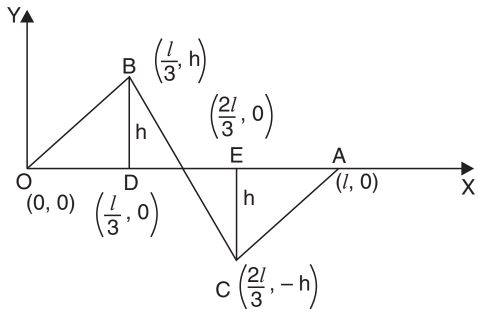
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad \text{...(2)}$$

Let l be the length of string (Refer Sol. of Ex. 1)

Eqn. of OB is,



$$y - 0 = \frac{h - 0}{\frac{l}{3} - 0}(x - 0)$$

$$\Rightarrow \qquad y = \frac{3h}{l} x \qquad \text{...(3)}$$

Eqn. of BC is,

$$y - h = \frac{-h - h}{\frac{2l}{3} - \frac{l}{3}}\left(x - \frac{l}{3}\right)$$

$$= \frac{-2h}{\left(\frac{l}{3}\right)}\left(x - \frac{l}{3}\right) = -\frac{6h}{l}\left(x - \frac{l}{3}\right)$$

$$y - h = -\frac{6hx}{l} + 2h$$

$$y = 3h - \frac{6hx}{l} = 3h\left(1 - \frac{2x}{l}\right) \qquad \text{...(4)}$$

Eqn. of CA is, $$y + h = \frac{0 + h}{l - \frac{2l}{3}}\left(x - \frac{2l}{3}\right) = \frac{3h}{l}\left(x - \frac{2l}{3}\right) = \frac{3hx}{l} - 2h$$

$$y = \frac{3hx}{l} - 3h = 3h\left(\frac{x}{l} - 1\right) \qquad \text{...(5)}$$

Hence Boundary conditions are

$$y(0, t) = 0, \;\; y(l, t) = 0$$

Initial conditions are

$$\frac{\partial y}{\partial t} = 0,$$

and
$$y(x, 0) = \begin{cases} \dfrac{3h}{l}x, & 0 \le x \le l/3 \\ \dfrac{3h}{l}(l-2x), & \dfrac{l}{3} \le x \le \dfrac{2l}{3} \\ \dfrac{3h}{l}(x-l), & \dfrac{2l}{3} \le x \le l \end{cases}$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0$.

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad ...(6)$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \; (n \in \text{I})$

$\therefore$ $p = \dfrac{n\pi}{l}$.

$\therefore$ From (6), $$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad ...(7)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0$

$\therefore$ From (7), $$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

The most general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad ...(8)$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3h}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3h}{l}(l-2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3h}{l}(x-l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\cdot\frac{3h}{l}\int_0^{l/3} x \sin \frac{n\pi x}{l}\, dx + \frac{2}{l}\cdot\frac{3h}{l}\int_{l/3}^{2l/3} (l-2x) \sin \frac{n\pi x}{l}\, dx$$

$$+ \frac{2}{l}\cdot\frac{3h}{l}\int_{2l/3}^{l} (x-l) \sin \frac{n\pi x}{l}\, dx$$

NOTES

$$= \frac{6h}{l^2}\left[\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{6h}{l^2}\left[\left\{(l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/3}^{2l/3} - \int_{l/3}^{2l/3}(-2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$+ \frac{6h}{l^2}\left[\left\{(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{2l/3}^{l} - \int_{2l/3}^{l} 1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{6h}{l^2}\left[\frac{-l}{n\pi}\cdot\frac{l}{3}\cos\frac{n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{-l}{3}\cdot\frac{l}{n\pi}\left(-\cos\frac{2n\pi}{3}\right) + \left(\cos\frac{n\pi}{3}\right)\frac{l}{3}\cdot\frac{l}{n\pi} - \frac{2l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/3}^{2l/3}\right]$$

$$+ \frac{6h}{l^2}\left[\frac{-l}{3}\cdot\frac{l}{n\pi}\cos\frac{2n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin n\pi x/l}{n\pi/l}\right)_{2l/3}^{l}\right]$$

$$= \frac{-2h}{n\pi}\cos\frac{n\pi}{3} + \frac{6h}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{2h}{n\pi}\cos\frac{2n\pi}{3}$$

$$+ \frac{2h}{n\pi}\cos\frac{n\pi}{3} - \frac{12h}{n^2\pi^2}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right) - \frac{2h}{n\pi}\cos\frac{2n\pi}{3} + \frac{6h}{n^2\pi^2}\left(0 - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\frac{2n\pi}{3} = \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{18h}{n^2\pi^2}\sin\left(n\pi - \frac{n\pi}{3}\right)$$

$$= \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{18h}{n^2\pi^2}\sin\frac{n\pi}{3}\cos n\pi$$

$$= \begin{cases} \frac{36h}{n^2\pi^2}\sin\frac{n\pi}{3}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

$\therefore$ From (8), $$y(x, t) = \frac{36h}{\pi^2}\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$y(x, t) = \frac{9h}{\pi^2}\sum_{m=1,2,\ldots}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\cos\frac{2m\pi ct}{l}\sin\frac{2m\pi x}{l} \qquad \ldots(9)$$

(where $n = 2m$)

Putting $x = \frac{l}{2}$ in eqn. (6), we get

$$y\left(\frac{l}{2}, t\right) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\left(\frac{2m\pi}{3}\right)\cdot\frac{1}{m^2}\cdot\cos\frac{2m\pi ct}{l}\cdot\sin m\pi = 0.$$

Hence, midpoint of the string is always at rest.

NOTES

Example 16. *If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity* $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, *find the displacement y (x, t).*

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of equation (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are, $y(0, t) = 0$ ...(3)

$y(l, t) = 0$ ...(4)

Initial conditions, are $y(x, 0) = 0$ ...(5)

and $\left(\frac{\partial y}{\partial t}\right) = b \sin^3 \frac{\pi x}{l}$ at $t = 0$ ...(6)

From eqn. (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \text{I})$

$\therefore$ $p = \frac{n\pi}{l}.$

$\therefore$ From (7), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ ...(8)

$y(x, 0) = 0 = c_1 c_4 \sin \frac{n\pi x}{l}$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (8), $y(x, t) = c_2 c_4 \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

$= b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$ where $c_2 c_4 = b_n$

The general solution is

$$y(x, t) = \sum_1^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad ...(9)$$

$$\frac{\partial y}{\partial t} = \sum_1^{\infty} b_n \cdot \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

At $t = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_1^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$

$$b \sin^3 \frac{\pi x}{l} = \sum_1^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2b_2 \pi c}{l} \sin \frac{2\pi x}{l} + 3b_3 \frac{\pi c}{l} \sin \frac{3\pi x}{l} + ...$$

NOTES

$\Rightarrow \qquad b_1 \dfrac{\pi c}{l} = \dfrac{3b}{4} \quad \Rightarrow \quad b_1 = \dfrac{3bl}{4\pi c}$

$b_2 = 0$ and $\dfrac{3b_3\pi c}{l} = -\dfrac{b}{4} \quad \Rightarrow \quad b_3 = -\dfrac{bl}{12\pi c}$

Also, $\qquad b_4 = 0 = b_5 = \ldots$ etc.

Hence from (9), $\quad y(x, t) = \dfrac{3bl}{4\pi c} \sin \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l} - \dfrac{bl}{12\pi c} \sin \dfrac{3\pi ct}{l} \sin \dfrac{3\pi x}{l}$

$$= \frac{bl}{12\pi c}\left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}\right].$$

Example 17. *A tightly stretched string with fixed end points x = 0 and x = l is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity λx(l – x), find the displacement of the string at any distance x from one end at any time t.*

Sol. Here the boundary conditions are $\quad y(0, t) = y(l, t) = 0$

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \qquad \ldots(1)$$

| Refer Sol. of Ex. 2

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore \quad$ From (1), $\quad 0 = \displaystyle\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad \Rightarrow \quad a_n = 0$

$$\therefore \qquad y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \ldots(2)$$

and
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But $\qquad \dfrac{\partial y}{\partial t} = \lambda x(l - x) \quad$ when $t = 0$

$$\therefore \qquad \lambda x(l - x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{\pi c}{l} nb_n = \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left[x(l - x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1 - (-1)^n]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases} \quad i.e., \ \frac{8\lambda l^2}{\pi^3 (2m - 1)^3}, \text{ taking } n = 2m - 1$$

$$\Rightarrow \quad b_n = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4}$$

$\therefore \quad$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi ct}{l} \sin \frac{(2m - 1)\pi x}{l}.$$

Example 18. *Solve:*

PDE $u_{tt} - u_{xx} = x - t, \; -\infty < x < \infty, \; t > 0$

ICs $u(x, 0) = x^4, \quad u_t(x, 0) = \sin x.$

Sol. Here we split the given problem into two problems with $u_1(x, t)$ and $u_2(x, t)$ and solve

$$(u_1)_{tt} = (u_1)_{xx}$$
$$(u_1)(x, 0) = x^4, \qquad (u_1)_t(x, 0) = \sin x$$

and
$$(u_2)_{tt} = (u_2)_{xx} + x - t$$
$$(u_2)(x, 0) = 0, \qquad (u_2)_t(x, 0) = 0$$

The solution of the given problem is then

$$u(x, t) = u_1(x, t) + u_2(x, t)$$

By D' Alembert's formula [$c = 1$, $f(x) = x^4$, $g(x) = \sin x$]

$$u_1(x, t) = \frac{1}{2}[(x + t)^4 + (x - t)^4] + \frac{1}{2}\int_{x-t}^{x+t} \sin\theta \, d\theta$$

$$= \frac{1}{2}[(x + t)^4 + (x - t)^4] - \frac{1}{2}[\cos(x + t) - \cos(x - t)]$$

$$= x^4 + 6x^2t^2 + t^4 + \sin x \sin t$$

By Duhamel's principle [$h(x, t) = x - t$]

$$u_2(x, t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} (r - s)\, dr ds = \frac{1}{2}\int_0^t \left[\frac{r^2}{2} - sr\right]_{x-(t-s)}^{x+(t-s)} ds$$

$$= \frac{1}{2}\int_0^t [2s^2 - 2(t + x)s + 2tx]\, ds = \left[\frac{s^3}{3} - (t + x)\frac{s^2}{2} + txs\right]_0^t$$

$$= \frac{t^3}{3} - (t + x)\frac{t^2}{2} + t^2x = -\frac{t^3}{6} + \frac{t^2x}{2}$$

Hence $(u, t) = x^4 + 6x^2t^2 + t^4 + \sin x \sin t - \dfrac{t^3}{6} + \dfrac{t^2x}{2}$.

EXERCISE B

1. Find the deflection $y(x, t)$ of the vibrating string of length π and ends fixed, corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x)$ given $c^2 = 1$.

2. Solve: $y_{tt} = 4y_{xx}$; $y(0, t) = 0 = y(5, t)$, $y(x, 0) = 0$ $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = f(x)$

if (*i*) $f(x) = 5 \sin \pi x$ (*ii*) $f(x) = 3 \sin 2\pi x - 2 \sin 5\pi x$.

3. Find the deflection of the vibrating string which is fixed at the ends $x = 0$ and $x = 2$ and the motion is started by displacing the string into the form $\sin^3\left(\dfrac{\pi x}{2}\right)$ and releasing it with zero initial velocity at $t = 0$.

4. Find the solution of the equation of a vibrating string of length l satisfying the initial conditions:

$$y = F(x) \qquad \text{when } t = 0$$

and
$$\frac{\partial y}{\partial t} = \phi(x) \qquad \text{when } t = 0$$

It is assumed that the equation of a vibrating string is $y_{tt} = a^2 y_{xx}$.

NOTES

5. The vibrations of an elastic string are governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is π and ends are fixed. The initial velocity is zero and the initial deflection is $u(x, 0) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string at any time t.

6. Using D' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection $f(x) = a(x - x^3)$.

7. The ends of a tightly stretched string of length l are fixed at $x = 0$ and $x = l$. The string is at rest with the point $x = b$ drawn aside through a small distance d and released at time $t = 0$. Show that

$$y(x, t) = \frac{2dl^2}{\pi^2 b(l-b)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi b}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}.$$

8. Find the deflection of the vibrating string of unit length whose end points are fixed if the initial velocity is zero and the initial deflection is given by $u(x, 0) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ -1, & \frac{1}{2} < x \le 1 \end{cases}$.

9. Find the deflection $u(x, t)$ of a tightly stretched vibrating string of unit length that is initially at rest and whose initial position is given by

$$\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x, \quad 0 \le x \le 1$$

10. A string is stretched and fastened to two points distance l apart. Find the displacement $y(x, t)$ at any point at a distance x from one end at time t given that:

$$y(x, 0) = A \sin\left(\frac{2\pi x}{l}\right)$$

11. Solve the following initial-boundary value problem (IBVP):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 2, \qquad u(1, t) = 3, \quad t \ge 0$$

$$u(x, 0) = 2 + x + \sin \pi x$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0.$$

12. Solve the following initial-boundary value problem (IBVP):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0$$

$$u_x(0, t) = 1, \qquad u_x(1, t) = 3, \quad t \ge 0$$

$$u(x, 0) = x + x^2, \qquad u_t(x, 0) = \pi \cos \pi x.$$

13. Use D'Alembert's formula to solve the IVP:

$$u_t = 4u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 4.$$

14. If $u(x, t) = \sin x \cos t + 2xt$ is a solution of the initial value problem

$$u_{tt} = u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 2x$$

Use D'Alembert's formula to find $f(x)$.

15. What is the solution to the initial-value problem:

(*i*) PDE $u_{tt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$

ICs $u(x, 0) = e^{-x^2}$, $u_t(x, 0) = 0$

(*ii*) PDE $u_{tt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$

ICs $u(x, 0) = 0$, $u_t(x, 0) = xe^{-x^2}$

(*iii*) PDE $u_{tt} = u_{xx} + 2$, $-\infty < x < \infty$, $t > 0$

ICs $u(x, 0) = 0$, $u_t(x, 0) = 0$

(*iv*) PDE $u_{tt} = 4u_{xx} + x \sin t$, $-\infty < x < \infty$, $t > 0$

ICs $u(x, 0) = 0$, $u_t(x, 0) = 0$.

Answers

1. $y(x, t) = k(\cos t \sin x - \cos 2t \sin 2x)$

2. (*i*) $y(x, t) = \dfrac{5}{2\pi} \sin \pi x \sin 2\pi t$ (*ii*) $y(x, t) = \dfrac{3}{4\pi} \sin 2\pi x \sin 4\pi t - \dfrac{1}{5\pi} \sin 5\pi x \sin 10\pi t$

3. $y(x, t) = \dfrac{3}{4} \sin \dfrac{\pi x}{2} \cos \dfrac{\pi ct}{2} - \dfrac{1}{4} \sin \dfrac{3\pi x}{2} \cos \dfrac{3\pi ct}{2}$

4. $y(x, t) = \sum_{n=1}^{\infty} \sin \dfrac{n\pi x}{l} \left(a_n \cos \dfrac{n\pi at}{l} + b_n \sin \dfrac{n\pi at}{l} \right)$

where $a_n = \dfrac{2}{l} \int_0^l F(x) \sin \dfrac{n\pi x}{l}\, dx$ and $b_n = \dfrac{2}{na\pi} \int_0^l \phi(x) \sin \dfrac{n\pi x}{l}\, dx$

5. $y(x, t) = 2[\cos t \sin x + \cos 3t \sin 3x]$ **6.** $y(x, t) = ax(1 - x^2 - 3c^2t^2)$

8. $y(x, t) = \dfrac{2}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \left(1 + \cos n\pi - 2\cos \dfrac{n\pi}{2} \right) \sin n\pi x \cos n\pi ct$

9. $u(x, t) = \sin \pi x \cos \pi ct + \dfrac{1}{3} \sin 3\pi x \cos 3\pi ct + \dfrac{1}{5} \sin 5\pi x \cos 5\pi ct$

10. $y(x, t) = A \sin \left(\dfrac{2\pi x}{l} \right) \cos \left(\dfrac{2\pi ct}{l} \right)$ **11.** $u(x, t) = 2 + x + \sin \pi x \cos \pi t$

12. $u(x, t) = x + x^2 + t^2 + \cos \pi x \sin \pi t$ **13.** $u(x, t) = \sin x \cos 2t + 4t$

14. $f(x) = \sin x$

15. (*i*) $u(x, t) = \dfrac{1}{2}[e^{-(x+t)^2} + e^{-(x-t)^2}]$ (*ii*) $u(x, t) = \dfrac{1}{4}[e^{-(x-t)^2} - e^{-(x+t)^2}]$

(*iii*) $u(x, t) = t^2$ (*iv*) $u(x, t) = x(t - \sin t)$

VIBRATING MEMBRANE—TWO-DIMENSIONAL WAVE EQUATION

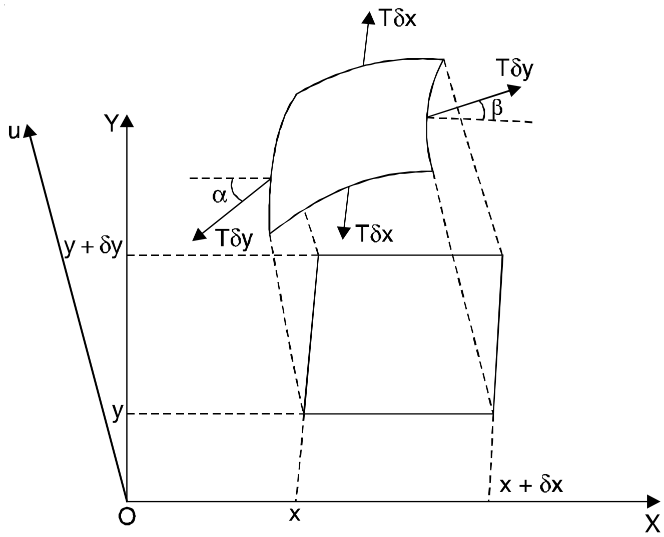
We shall now obtain the equation for the vibrations of a tightly stretched membrane (such as the membrane of a drum). We shall assume that the membrane is uniform and the tension in it per unit length is the same at every point in all directions. Let T be the tension per unit length and m be the mass of the membrane per unit area.

Consider the forces on an element $\delta x\delta y$ of the membrane. Due to its displacement u, perpendicular to the xy-plane, the forces $T\delta y$ (tangential to the membrane) on its opposite edges of length δy act at angles α and β to the horizontal. So their vertical component

$= (T\delta y)\sin\beta - (T\delta y)\sin\alpha = T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

$$= T\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] = T\delta y\delta x\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

$= T\delta x\delta y\,\dfrac{\partial^2 u}{\partial x^2}$ upto a first order of approximation.



Similarly, the forces $T\delta x$ acting on the edges of length δx have the vertical component $T\delta x\delta y\,\dfrac{\partial^2 u}{\partial y^2}$.

Hence the equation of motion of the element is

$$(m\delta x\delta y)\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\delta x\delta y$$

or $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)} \text{ where } c^2 = \frac{T}{m}.$$

This is the wave equation in two dimensions.

SOLUTION OF TWO-DIMENSIONAL WAVE EQUATION

Two-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ...(1)$$

Let $u = XYT$ ...(2)

where X is a function of x only, Y is a function of y only and T is a function of t only, be a solution of (1).

Then $\frac{\partial^2 u}{\partial t^2} = XYT''$, $\frac{\partial^2 u}{\partial x^2} = X''YT$ and $\frac{\partial^2 u}{\partial y^2} = XY''T$

Substituting in (1), we have $\frac{1}{c^2} XYT'' = X''YT + XY''T$

Dividing by XYT, we have $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$ ...(3)

This will be true only when *each member is a constant.* Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)\, c^2T = 0$$

The solutions of these equations are respectively

$$X = c_1 \cos kx + c_2 \sin kx$$

$$Y = c_3 \cos ly + c_4 \sin ly$$

and

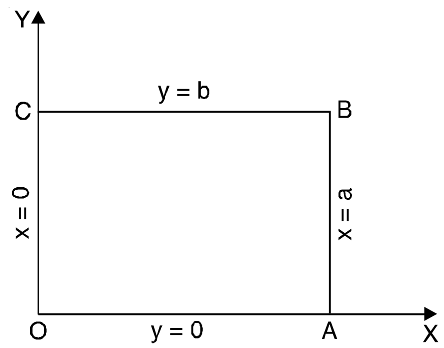
$$T = c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 + l^2)}\, ct$$

Hence from (2), a solution of (1) is

$$u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)$$
$$[c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 + l^2)}\, ct] \qquad ...(4)$$

Now let us suppose that the membrane is rectangular and is stretched between the lines

$$x = 0, x = a, y = 0, y = b.$$



Then the boundary conditions are:

(*i*) $u = 0$, when $x = 0$

(*ii*) $u = 0$, when $x = a$

(*iii*) $u = 0$, when $y = 0$

(*iv*) $u = 0$, when $y = b$ for all t.

Applying the condition (*i*), we have

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)}\, ct + c_6 \sin \sqrt{(k^2 + l^2)}\, ct]$$

i.e., $c_1 = 0$

Putting $c_1 = 0$ in (3) and applying the condition (*ii*), we have $\sin ka = 0$ or $k = \frac{m\pi}{a}$, where m is an integer.

Similarly, applying the conditions (*iii*) and (*iv*), we get

$$c_3 = 0 \quad \text{or} \quad l = \frac{n\pi}{b}, \text{ where } n \text{ is an integer.}$$

Therefore, the solution (3) becomes

$$u(x, y, t) = c_2c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \ (c_5 \cos pt + c_6 \sin pt)$$

NOTES

where $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$

Replacing the arbitrary constants suitably, we can write the general solution as

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \ (A_{mn} \cos pt + B_{mn} \sin pt) \qquad ...(5)$$

Now suppose the membrane starts from rest from the initial position $u = f(x, y)$ *i.e.*, $u(x, y, 0) = f(x, y)$.

Then applying the condition: $\frac{\partial u}{\partial t} = 0$ when $t = 0$, we get $B_{mn} = 0$.

Also using the condition: $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad ...(6)$$

This is a double Fourier series. Multiplying both sides by $\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one becomes zero. Thus, we get

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dy \, dx = \frac{ab}{4} A_{mn}$$

i.e.,

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{a} \, dy \, dx \qquad ...(7)$$

Hence, from (5), the required solution is

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where A_{mn} is given by (7) and $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

SOLVED EXAMPLES

Example 19. *Find the deflection u(x, y, t) of the square membrane with a = b = c = 1, if the initial velocity is zero and the initial deflection f(x, y) = A sin πx sin 2πy.*

Sol. The vibrations of the square membrane are governed by two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad ...(1)$$

Here the boundary conditions are,

$$u(0, y, t) = 0, \ u(1, y, t) = 0, \quad u(x, 0, t) = 0, \ u(x, 1, t) = 0$$

and the initial conditions are

$$u(x, y, 0) = f(x, y) = A \sin \pi x \sin 2\pi y \quad \text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0$$

To solve eqn. (1), let $u = XYT$ ...(2)
where X is function of x only, Y is a function of y only, and T is a function of t only.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial t^2}(XYT) = XY\frac{d^2T}{dt^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XYT) = YT\frac{d^2X}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XYT) = XT\frac{d^2Y}{dy^2},$$

From (1), $XYT'' = (YTX'' + XTY'')c^2$

$$\Rightarrow \quad \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

This will be true only when each member is a constant. Choosing the constant suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)\,c^2T = 0.$$

The solutions of these equations are respectively,

$$X = c_1 \cos kx + c_2 \sin kx$$

$$Y = c_3 \cos ly + c_4 \sin ly$$

and $$T = c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct.$$

$$\therefore \quad u(x, y, t) = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin y)$$
$$[c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct] \quad ...(3)$$

From (3), $u(0, y, t) = 0 = c_1(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct)$

$$\Rightarrow \quad c_1 = 0.$$

$\therefore$ From (3),

$$u(x, y, t) = c_2 \sin kx(c_3 \cos ly + c_4 \sin ly)$$
$$(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct) \quad ...(4)$$

$$u(1, y, t) = 0 = c_2 \sin k(c_3 \cos ly + c_4 \sin ly)(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct)$$

$$\Rightarrow \quad \sin k = 0 = \sin m\pi \ (m \in I)$$

$$k = m\pi.$$

$\therefore$ From (4), $u(x, y, t) = c_2 \sin m\pi x(c_3 \cos ly + c_4 \sin ly)$

$$(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct) \quad ...(5)$$

$$u(x, 0, t) = 0 = c_2 \sin m\pi x \,.\, c_3(c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct]$$

$$\Rightarrow \quad c_3 = 0.$$

$\therefore$ From (5),

$$u(x, y, t) = c_2c_4 \sin m\pi x \sin ly\ (c_5 \cos \sqrt{k^2 + l^2}\, ct + c_6 \sin \sqrt{k^2 + l^2}\, ct) \quad ...(6)$$

NOTES

$$u(x, 1, t) = 0 = c_2c_4 \sin m\pi x \sin l\,(c_5 \cos \sqrt{k^2+l^2}\,ct + c_6 \sin \sqrt{k^2+l^2}\,ct)$$

$\Rightarrow$ $\sin l = 0 = \sin n\pi \; (n \in \text{I})$

$\Rightarrow$ $l = n\pi.$

$\therefore$ From (6),

$$u(x, y, t) = c_2c_4 \sin m\pi x \sin n\pi y\,(c_5 \cos \sqrt{k^2+l^2}\,ct + c_6 \sin \sqrt{k^2+l^2}\,ct)$$

$$u(x, y, t) = \sin m\pi x \sin n\pi y\,(\text{A}_{mn} \cos pt + \text{B}_{mn} \sin pt) \quad ...(7)$$

where $p = \pi c\sqrt{m^2+n^2}$.

The most general solution is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin m\pi x \sin n\pi y\,(\text{A}_{mn} \cos pt + \text{B}_{mn} \sin pt) \quad ...(8)$$

Applying the condition,

$$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0$$

we get $\text{B}_{mn} = 0$ ...(9)

Also, using the condition $u = f(x, y) = \text{A} \sin \pi x \sin 2\pi y$ when $t = 0$, we get

$$\text{A} \sin \pi x \sin 2\pi y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \text{A}_{mn} \sin m\pi x \sin n\pi y$$

This is a double Fourier Series.

$$\text{A}_{mn} = \frac{2}{1}\cdot\frac{2}{1}\int_0^1\int_0^1 \text{A} \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y\, dx\, dy$$

Obviously, $\text{A}_{m1} = \text{A}_{m3} = \text{A}_{m4} = ... = 0$

But,
$$\text{A}_{m2} = 4\text{A}\int_0^1\int_0^1 \sin \pi x \sin m\pi x \sin^2 2\pi y\, dx\, dy$$
$$= 2\text{A}\int_0^1\int_0^1 \sin \pi x \sin m\pi x\,(1 - \cos 4\pi y)\, dx\, dy$$
$$= 2\text{A}\int_0^1 \sin \pi x \sin m\pi x \left(y - \frac{\sin 4\pi y}{4\pi}\right)_0^1 dx$$
$$= 2\text{A}\int_0^1 \sin \pi x \sin m\pi x\, dx$$

Again, obviously,

$$\text{A}_{22} = \text{A}_{32} = \text{A}_{42} = ... = 0$$

But,
$$\text{A}_{12} = 2\text{A}\int_0^1 \sin^2 \pi x\, dx = \text{A}\int_0^1 (1 - \cos 2\pi x)\, dx$$
$$= \text{A}\left(x - \frac{\sin 2\pi x}{2\pi}\right)_0^1 = \text{A} \quad ...(10)$$

$\therefore$ From (8), (9) and (10), we get

$$u(x, y, t) = \text{A} \sin \pi x \sin 2\pi y \cos pt \quad ...(11)$$

where $p = \pi c\sqrt{m^2+n^2} = \pi(1)\sqrt{1+4} = \pi\sqrt{5}$.

$\therefore$ From (11), $u(x, y, t) = A \cos \pi\sqrt{5}\, t \sin \pi x \sin 2\pi y$.

Example 20. *Find the deflection u(x, y, t) of a rectangular membrane $(0 \le x \le a,\ 0 \le y \le b)$ whose boundary is fixed ; given that it starts from rest and $u(x, y, 0) = xy(a - x)(b - y)$. Show that the deflection u is given by*

NOTES

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $$A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6}(1 - \cos m\pi)(1 - \cos n\pi) \quad \text{and} \quad k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Sol. Proceeding as in Art. 2.11, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad ...(1)$$

From (1), $$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$xy(a - x)(b - y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b xy(a - x)(b - y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dy\, dx$$

$$= \frac{4}{ab} \int_0^a x(a - x) \sin \frac{m\pi x}{a}\, dx \cdot \int_0^b y(b - y) \sin \frac{n\pi y}{b}\, dy$$

$$= \frac{4}{ab}\left[x(a - x)\left(\frac{-\cos \frac{m\pi x}{a}}{\frac{m\pi}{a}}\right) - (a - 2x)\left(\frac{-\sin \frac{m\pi x}{a}}{\frac{m^2\pi^2}{a^2}}\right) + (-2)\left(\frac{\cos \frac{m\pi x}{a}}{\frac{m^3\pi^3}{a^3}}\right)\right]_0^a$$

$$\times \left[y(b - y)\left(\frac{-\cos \frac{n\pi y}{b}}{\frac{n\pi}{b}}\right) - (b - 2y)\left(\frac{-\sin \frac{n\pi y}{b}}{\frac{n^2\pi^2}{b^2}}\right) + (-2)\left(\frac{\cos \frac{n\pi y}{b}}{\frac{n^3\pi^3}{b^3}}\right)\right]_0^b$$

$$= \frac{4}{ab}\left[\frac{-2a^3}{m^3\pi^3} \cos m\pi + \frac{2a^3}{m^3\pi^3}\right]\left[\frac{-2b^3}{n^3\pi^3} \cos n\pi + \frac{2b^3}{n^3\pi^3}\right]$$

$$= \frac{4}{ab} \cdot \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} [1 - (-1)^n][1 - (-1)^m]$$

$$A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6} [1 - (-1)^n][1 - (-1)^m].$$

Hence from (1),

$$u(x, y, t) = \frac{16a^2b^2}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n][1 - (-1)^m]}{m^3n^3} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

NOTES

where $$p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad ...(2)$$

Expression (2) may also be put in as,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where $$A_{mn} = \frac{16a^2b^2}{m^3\pi^6n^3} (1 - \cos m\pi)(1 - \cos n\pi) \quad \text{and} \quad k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Example 21. *A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is k sin 2πx sin πy. Show that the deflection at any instant is*

$$k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct).$$

Sol. Here we have to solve the equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

with boundary conditions $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$

and the initial conditions $u(x, y, 0) = f(x, y) = k \sin 2\pi x \sin \pi y$

$$\frac{\partial u}{\partial t} = 0 \quad \text{when } t = 0$$

Proceeding as in Ex. 1, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y \cos pt \qquad ...(1)$$

Since, $a = b = 1$, where $p = \pi c \sqrt{m^2 + n^2}$

and
$$A_{mn} = \frac{4}{1 \times 1} \int_0^1 \int_0^1 k \sin 2\pi x \sin \pi y \sin m\pi x \sin n\pi y \, dy dx$$
$$= 4k \int_0^1 \sin m\pi x \sin 2\pi x \, dx \int_0^1 \sin n\pi y \sin \pi y \, dy$$
$$= 0 \quad \text{for } m \neq 2 \quad \text{or} \quad n \neq 1$$

$\therefore$ $$A_{21} = 4k(\tfrac{1}{2})(\tfrac{1}{2}) = k \quad \text{and} \quad p = \pi c \sqrt{(2)^2 + (1)^2} = \sqrt{5}\, \pi c$$

Hence solution (1) reduces to $u(x, y, t) = k \sin 2\pi x \sin \pi y \cos (\sqrt{5}\, \pi ct)$.

EXERCISE C

1. Find the deflecting $u(x, y, t)$ of a rectangular membrane $(0 < x < 1,\ 0 < y < 2)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy(1 - x)(2 - y)$.
2. Find the deflection $u(x, y, t)$ of a rectangular membrane $(0 < x < a,\ 0 < y < b)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = xy(a^2 - x^2)(b^2 - y^2)$.
3. Find the deflection $u(x, y, t)$ of the tightly stretched rectangular membrane with sides a and b having wave velocity $c = 1$ if the initial velocity is zero and the initial deflection is
$$f(x, y) = \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b}.$$

NOTES

Answers

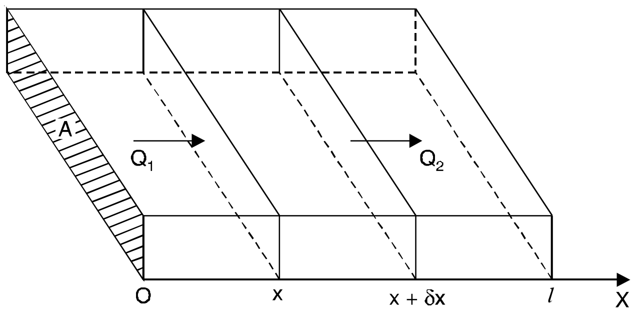
1. $u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin \frac{n\pi y}{2} \cos pt$, where $A_{mn} = \frac{256}{m^3 n^3 \pi^6}$, both m, n odd and $p = \pi c\sqrt{m^2 + \frac{n^2}{4}}$

2. $u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$, where $A_{mn} = \frac{144a^3b^3}{m^3 n^3 \pi^6}$ and $p = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$.

3. $u(x, y, t) = \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b} \cos pt$ where $p = \pi\sqrt{\frac{4}{a^2} + \frac{9}{b^2}}$.

ONE-DIMENSIONAL HEAT FLOW $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Consider the flow of heat by conducting in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible. Take one end of the bar as origin and the direction of flow as the positive *x*-axis. The temperature *u* at any point of the bar depends on the distance *x* of the point from one end and the time *t*. Also, the temperature of all points of any cross-section is the same.



The amount of heat crossing any section of the bar per second depends on the area A of the cross-section, the conductivity K of the material of the bar and the temperature gradient $\frac{\partial u}{\partial x}$ *i.e.*, rate of change of temperature w.r.t. distance normal to the area.

$\therefore$ Q_1, the quantity of heat flowing into the section at a distance x

$$= -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per sec.}$$

(the negative sign on the right is attached because as x increases, u decreases).

Q_2, the quantity of heat flowing out of the section at a distance $x + \delta x$

$$= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per sec.}$$

NOTES

Hence the amount of heat retained by the slab with thickness δx is

$$Q_1 - Q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \text{ per sec.} \qquad ...(1)$$

But the rate of increase of heat in the slab = $S\rho A\,\delta x\,\frac{\partial u}{\partial t}$ ...(2)

where S is the specific heat and ρ, the density of the material.

$$\therefore \quad \text{From (1) and (2), } S\rho A\,\delta x\,\frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

or
$$S\rho\,\frac{\partial u}{\partial t} = K\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right]$$

Taking the limit as $\delta x \to 0$, we have

$$S\rho\,\frac{\partial u}{\partial t} = K\,\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial u}{\partial t} = \frac{K}{S\rho}\,\frac{\partial^2 u}{\partial x^2}$$

or $\boxed{\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}}$, where $c^2 = \frac{K}{S\rho}$ is known as diffusivity of the material of the bar.

SOLUTION OF THE HEAT EQUATION

The heat equation is
$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Let
$$u = XT \qquad ...(2)$$

where X is a function of x only and T is a function of t only, be a solution of (1).

Then
$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting in (1), we have $XT' = c^2X''T$

Separating the variables, we get
$$\frac{X''}{X} = \frac{1}{c^2}\,\frac{T'}{T} \qquad ...(3)$$

Now the LHS of (3) is a function of x only and the R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant, say k. The equation (3) leads to the ordinary differential equations.

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{dT}{dt} - kc^2T = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1e^{px} + c_2e^{-px},\; T = c_3\,e^{c^2p^2t}$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px,\; T = c_3\,e^{-c^2p^2t}$$

NOTES

(*iii*) *When* $k = 0$

$$X = c_1x + c_2,\ T = c_3.$$

Thus the various possible solutions of the heat equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px}) \cdot c_3 e^{c^2p^2t} \qquad ...(5)$$

$$u = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-c^2p^2t} \qquad ...(6)$$

$$u = (c_1x + c_2)\, c_3 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Since u decreases as time t increases, the only suitable solution of the heat equation is

$$u = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t}.$$

Solution (5) is rejected since $u \to \infty$ as $t \to \infty$.

Also, solution (7) is rejected as it gives a non-zero temperature at all times.

INHOMOGENEOUS BOUNDARY CONDITION

Now consider the case where the ends of a rod are kept at constant temperatures different from zero.

Consider the IBVP, $u_t = u_{xx}$, $0 < x < L$, $t > 0$

$$u(0, t) = \alpha,\ u(L, t) = \beta;\ t \geq 0$$

$$u(x, 0) = f(x)$$

To convert the inhomogeneous boundary conditions to homogeneous boundary conditions, we use the following transformation formula:

$$u(x, t) = \left[\alpha + \left(\frac{\beta - \alpha}{L}\right)x\right] + v(x, t)$$

We can easily show that now $v(x, t)$ will be governed by the IBVP:

$$v_t = v_{xx}, \qquad 0 < x < L, \qquad t > 0$$

$$v(0, t) = 0 = v(L, t), \qquad t \geq 0$$

$$v(x, 0) = f(x) - \left[\alpha + \left(\frac{\beta - \alpha}{L}\right)x\right]$$

$v(x, t)$ can now easily be found using method of separation of variables. Consequently $u(x, t)$ can readily be obtained as a final result.

SOLVED EXAMPLES

Example 1. *A rod of length l with insulated sides is initially at a uniform temperature u_0. Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function u(x, t).*

Sol. The temperature function $u(x, t)$ satisfies the differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

As proved in Art. 2.13, we have

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t} \quad ...(1)$$

Since the ends $x = 0$ and $x = l$ are cooled to 0°C and kept at that temperature throughout, the boundary conditions are $u(0, t) = u(l, t) = 0$ for all t

Also $u(x, 0) = u_0$ is the initial condition.

Since $u(0, t) = 0$, we have from (1),

$$0 = c_1 c_3 e^{-c^2p^2t} \quad \Rightarrow \quad c_1 = 0$$

$\therefore$ From (1), $u(x, t) = c_2 c_3 \sin px .\, e^{-c^2p^2t}$ ...(2)

Since $u(l, t) = 0$, we have from (2),

$$0 = c_2 c_3 \sin pl \,.\, e^{-c^2p^2t}$$

$$\Rightarrow \quad \sin pl = 0 \quad \Rightarrow \quad pl = n\pi$$

$$\therefore \quad p = \frac{n\pi}{l}, \; n \text{ being an integer}$$

Solution (2) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} . e^{\frac{-c^2n^2\pi^2t}{l^2}}$ on replacing c_2c_3 by b_n.

The most general solution is obtained by adding all such solutions for $n = 1, 2, 3, ...$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . e^{\frac{-c^2n^2\pi^2t}{l^2}} \quad ...(3)$$

Since $u(x, 0) = u_0$, we have $u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

which is half-range sine series for u_0.

$$b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l}\, dx = \begin{cases} 0 \;, & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi} \;, & \text{when } n \text{ is odd} \end{cases}$$

Hence the temperature function

$$u(x, t) = \frac{4u_0}{\pi} \sum_{n=1, 3, 5 ......}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{c^2n^2\pi^2t}{l^2}}$$

$$= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l}\, e^{-\frac{c^2(2n-1)^2\pi^2t}{l^2}} .$$

Example 23. *Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is sin* $\frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$.

Sol. Let $u(x, t)$ be the temperature in the bar. The boundary conditions are

$$u(0, t) = 0 = u(2, t) \text{ for any } t. \quad ...(1)$$

The initial condition is

$$u(x, 0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} \quad ...(2)$$

One-dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(3)$$

Its solution is

$$u(x, t) = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t} \qquad ...(4)$$

$$u(0, t) = 0 = c_1c_3 e^{-c^2p^2t} \qquad | \text{ Using (1)}$$

$\Rightarrow \qquad c_1 = 0$

$\therefore$ From (4),

$$u(x, t) = c_2c_3 \sin px\; e^{-c^2p^2t} \qquad ...(5)$$

$$u(2, t) = 0 = c_2c_3 \sin 2p\, e^{-c^2p^2t} \qquad | \text{ Using (1)}$$

$\Rightarrow \qquad \sin 2p = 0 = \sin n\pi$

$\therefore \qquad p = \dfrac{n\pi}{2},\ n \in \text{I}$

Hence from (5),

$$u(x, t) = b_n \sin \frac{n\pi x}{2}\; e^{-\frac{n^2\pi^2c^2t}{4}} \qquad | \because \ c_2c_3 = b_n$$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}\; e^{-\frac{n^2\pi^2c^2t}{4}} \qquad ...(6)$$

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + ... + b_5 \sin\left(\frac{5\pi x}{2}\right) + ...$$

Comparing, we get

$$b_1 = 1 \text{ and } b_5 = 3$$

Hence from (6),

$$u(x, t) = \sin\left(\frac{\pi x}{2}\right) e^{-\pi^2c^2t/4} + 3\sin\left(\frac{5\pi x}{2}\right) e^{-25\pi^2c^2t/4}.$$

Example 24. *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

Sol. Initial temperature distribution in the rod is

$$u_1 = 0 + \left(\frac{100-0}{l}\right)x = \frac{100}{l}x$$

Final temperature distribution (*i.e.,* in steady state) is

$$u_2 = 0 + \left(\frac{0-0}{l}\right)x = 0$$

To get u in the intermediate period,

$$u = u_2(x) + u_1(x, t)$$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$u_1(x, t)$ satisfies one dimensional heat flow equation

$$\therefore \qquad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2p^2t} \qquad \ldots(1)$$

In steady state, $u(0, t) = 0 = u(l, t)$ ...(2)

$$\therefore \text{ From (1), } \quad u(0, t) = 0 = \sum_{n=1}^{\infty} a_n e^{-c^2p^2t} \quad \Rightarrow \quad a_n = 0 \qquad \ldots(3)$$

$$\text{Also,} \quad u(l, t) = 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t} \qquad |\text{ using (3)}$$

$$\Rightarrow \quad \sin pl = 0 = \sin n\pi,\ n \in \text{I}$$

or

$$p = \frac{n\pi}{l} \qquad \ldots(4)$$

$$\therefore \text{ From (1), (3) and (4), } u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t} \qquad \ldots(5)$$

Using initial condition,

$$u(x, 0) = \frac{100}{l}x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is half-range sine series for $\frac{100}{l}x$.

$$\therefore \quad b_n = \frac{2}{l}\int_0^l \frac{100}{l} x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[\left\{x \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l 1 \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{200}{l^2}\left[\frac{-l^2}{n\pi}\cos n\pi + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right] = \frac{-200}{n\pi}(-1)^n$$

Hence the temperature function

$$u(x, t) = -\frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

In the second part, the initial condition remains the same as in first part *i.e.,*

$$u(x, 0) = \frac{100}{l}x.$$

NOTES

Boundary conditions are $u(0, t) = 20$ and $u(l, t) = 80$ for all values of t then, final temperature distribution is

$$u_2 = 20 + \left(\frac{80 - 20}{l}\right)x = 20 + \frac{60}{l}x$$

Then, $u = u_2(x) + u_1(x, t)$

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2p^2t} \qquad \ldots(6)$$

$$u(0, t) = 20 = 20 + \sum_{n=1}^{\infty} a_n e^{-c^2p^2t} \qquad | \text{ From (6)}$$

$\Rightarrow$ $a_n = 0$

$\therefore$ From (6),

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin px\, e^{-c^2p^2t} \qquad \ldots(7)$$

$$u(l, t) = 80 = 20 + \frac{60}{l}l + \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t} \qquad | \text{ From (7)}$$

$$\Rightarrow \quad 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t}$$

$$\sin pl = 0 = \sin n\pi,\ n \in \mathrm{I}$$

$$\therefore \quad p = \frac{n\pi}{l} \qquad \ldots(8)$$

From (7) and (8),

$$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t} \qquad \ldots(9)$$

Using initial condition,

$$u(x, 0) = \frac{100}{l}x = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \quad \frac{40}{l}x - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l}\int_0^l \left(\frac{40}{l}x - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{\left(\frac{40}{l}x - 20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l \frac{40}{l}\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right]$$

$$= \frac{-40}{n\pi}(1 + \cos n\pi) = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \dfrac{-80}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

NOTES

Hence equation (9) becomes,

$$u(x, t) = 20 + \frac{60}{l}x - \frac{80}{\pi} \sum_{\substack{n = 2, 4, \ldots \\ (n \text{ is even})}}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l} e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= 20 + \frac{60}{l}x - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} e^{-\frac{4c^2m^2\pi^2 t}{l^2}} . \text{ (taking } n = 2m)$$

Example 25. *The ends A and B of a rod of length 20 cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function u(x, t). The specific heat, density and the thermal conductivity of the material of the rod are such that the combination* $\frac{k}{\rho\sigma} = c^2 = 1$.

Sol. Initial temperature distribution in the rod is

$$u_1 = 30 + \left(\frac{80 - 30}{20}\right) x = 30 + \frac{5}{2} x$$

Final temperature distribution (*i.e.,* in steady state) is

$$u_2 = 40 + \left(\frac{60 - 40}{20}\right) x = 40 + x$$

To get u in the intermediate period,

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

∵ $u_1(x, t)$ satisfies one dimensional heat flow equation.

$$\therefore \quad u = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px) e^{-p^2 t} \quad \text{...(1)}$$

In steady state,

$$u(0, t) = 40 \quad \text{...(2)}$$

$$u(20, t) = 60 \quad \text{...(3)}$$

From (1), and (2), $\quad u(0, t) = 40 = 40 + \sum_1^{\infty} a_n e^{-p^2 t} \quad$ | From (2)

$$0 = \sum_1^{\infty} a_n e^{-p^2 t} \Rightarrow a_n = 0 \quad \text{...(4)}$$

∴ From (1),

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin px \, e^{-p^2 t}$$

Again, $\quad u(20, t) = 60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20p \, e^{-p^2 t}$

$$\Rightarrow \quad 0 = \sum_1^{\infty} b_n \sin 20p \, e^{-p^2 t}$$

$$\sin 20p = 0 = \sin n\pi, \; n \in \text{I}$$

NOTES

$\Rightarrow \qquad p = \dfrac{n\pi}{20}$

$\therefore \qquad u = 40 + x + \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t}$ ...(5)

Using initial condition,

$$u(x, 0) = 30 + \frac{5}{2}x \quad \text{in eqn. (5), we get}$$

$\Rightarrow \qquad 30 + \dfrac{5}{2}x = 40 + x + \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

$\Rightarrow \qquad \dfrac{3}{2}x - 10 = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$

where $\qquad b_n = \dfrac{2}{20}\displaystyle\int_0^{20} \left(\frac{3}{2}x - 10\right) \sin \frac{n\pi x}{20}\, dx = -\frac{20}{n\pi}\, [2(-1)^n + 1]$

From (5), $\qquad u(x, t) = 40 + x - \dfrac{20}{\pi} \displaystyle\sum_{n=1}^{\infty} \left\{\frac{2(-1)^n + 1}{n}\right\} \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t}$.

Example 26. *The temperature distribution in a bar of length π which is perfectly insulated at ends x = 0 and x = π is governed by partial differential equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Assuming the initial temperature distribution as u(x, 0) = f(x) = cos 2x, find the temperature distribution at any instant of time.

Sol. $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ ...(1)

Its solution is $\qquad u(x, t) = c_1 e^{-p^2 t}\, (c_2 \cos px + c_3 \sin px)$ ...(2)

Since ends of bar are insulated, no heat can pass from either sides and boundary conditions are

$\qquad \dfrac{\partial u}{\partial x} = 0 \quad \text{at } x = 0$ ...(3)

and $\qquad \dfrac{\partial u}{\partial x} = 0 \quad \text{at } x = \pi$ ..(4)

From (2), $\qquad \dfrac{\partial u}{\partial x} = c_1 e^{-p^2 t}\, (-pc_2 \sin px + pc_3 \cos px)$

At $x = 0$, $\qquad 0 = c_1 e^{-p^2 t}\, pc_3 \quad \Rightarrow \quad c_3 = 0$

$\therefore$ From (2), $\quad u(x, t) = c_1 c_2\, e^{-p^2 t} \cos px$ ...(5)

Again $\qquad \dfrac{\partial u}{\partial x} = -pc_1 c_2\, e^{-p^2 t} \sin px$

At $x = \pi$,

$$0 = -pc_1 c_2 e^{-p^2 t} \sin p\pi$$

$\Rightarrow \qquad \sin p\pi = 0 = \sin n\pi \ (n \in \text{I})$

$$p\pi = n\pi \quad \Rightarrow \quad p = n$$

$\therefore$ From (5), $\quad u(x, t) = b_n\, e^{-n^2 t} \cos nx, \quad$ where $\quad c_1 c_2 = b_n$

Most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \cos nx \qquad ...(6)$$

$$u(x, 0) = \cos 2x = \sum_{n=1}^{\infty} b_n \cos nx$$

Comparing, we get $b_2 = 1$ and $n = 2$. All other b_i's are zero.

∴ From (6), $u(x, t) = e^{-4t} \cos 2x$.

Example 27. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary condition* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(l, t) = 0$, *where* $0 < x < l$.

Sol. The solution to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

is given by $u(x, t) = c_1 e^{-p^2 t} (c_2 \cos px + c_3 \sin px)$ ...(2)

From (2), $u(0, t) = c_1 c_2 e^{-p^2 t}$

$\Rightarrow \quad 0 = c_1 c_2 e^{-p^2 t}$

$\Rightarrow \quad c_2 = 0.$

∴ From (2), $u(x, t) = c_1 c_3 e^{-p^2 t} \sin px$ ...(3)

$u(l, t) = 0 = c_1 c_3 e^{-p^2 t} \sin pl$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi (n \in \text{I})$

$\therefore \quad p = \frac{n\pi}{l}.$

From (3), $u(x, t) = c_1 c_3 e^{-\frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi x}{l} = b_n e^{-\frac{n^2\pi^2 t}{l^2}} \sin \frac{n\pi x}{l}$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t/l^2)} \sin \frac{n\pi x}{l} \qquad ...(4)$$

∴ From (4), $u(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$\Rightarrow \quad 3 \sin n\pi x = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l}.$

Comparison gives, $b_n = 3$, $l = 1$.

Hence from (4), the required solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 28. *A bar with insulated sides is initially at a temperature 0°C throughout. The end x = 0 is kept at 0°C, and heat is suddenly applied at the end x = l so that* $\frac{\partial u}{\partial x} = A$ *for x = l, where A is a constant. Find the temperature function u(x, t).*

Sol. One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

Its solution is

$$u(x, t) = c_1 e^{-p^2c^2t} (c_2 \cos px + c_3 \sin px)$$

or $$u(x, t) = (A_1 \cos px + B \sin px)\, e^{-p^2c^2t} \quad ...(2)$$

Applying the zero end conditions as,

$$u(0, t) = 0 = A_1 e^{-p^2c^2t}$$

$\Rightarrow$ $A_1 = 0.$

$\therefore$ From (2), $$u(x, t) = B \sin px e^{-p^2c^2t} \quad ...(3)$$

From (3), $$\frac{\partial u}{\partial x} = pB \cos px e^{-p^2c^2t}.$$

At $x = l$, $$\left(\frac{\partial u}{\partial x}\right)_{x=l} = 0 = pB \cos pl e^{-p^2c^2t}$$

$\Rightarrow$ $$\cos pl = 0 = \cos\left(n\pi - \frac{\pi}{2}\right) ;\ n \in I \quad \text{or} \quad pl = (2n-1)\frac{\pi}{2}$$

$\Rightarrow$ $$p = (2n-1)\frac{\pi}{2l}.$$

$\therefore$ From (3), $$u(x, t) = B \sin px e^{-p^2c^2t} \quad ...(4) \quad \text{where } p = (2n-1)\frac{\pi}{2l}$$

The most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin px\, e^{-p^2c^2t} \quad ...(5) \quad \text{where } p = (2n-1)\frac{\pi}{2l}.$$

The end conditions given for this problem are

(*i*) $u = 0$ at $x = 0$ (*ii*) $\frac{\partial u}{\partial x} = A$ at $x = l$ ...(6)

These conditions are different from the zero end conditions. So we add to (5) the solution

$$u = A_1 x + B$$

Choosing A_1 and B so that (6) is satisfied.

This gives, $0 = B$ and $A_1 = A$

$\therefore$ $$u(x, t) = Ax + \sum_{n=1}^{\infty} B_n \sin px\, e^{-p^2c^2t} \quad ...(7) \quad \text{where } p = (2n-1)\frac{\pi}{2l}.$$

Applying the condition that $u = 0$ at $t = 0$, we have

$$0 = Ax + \sum_{n=1}^{\infty} B_n \sin px$$

or $$-Ax = \sum_{n=1}^{\infty} B_n \sin px$$

where $$B_n = \frac{2}{l}\int_0^l (-Ax)\sin px\,dx, \text{ where } p = (2n-1)\frac{\pi}{2l}$$

$$= \frac{-2A}{l}\left[\left\{x\left(\frac{-\cos px}{p}\right)\right\}_0^l - \int_0^l 1.\left(\frac{-\cos px}{p}\right)dx\right]$$

$$= -\frac{2A}{l}\left[\frac{-l\cos pl}{p} + \frac{1}{p}\left(\frac{\sin px}{p}\right)_0^l\right]$$

$$= \frac{-2A}{l}\left[\frac{-l\cos pl}{p} + \frac{1}{p^2}\sin pl\right] = -\frac{2A(2l)^2}{l(2n-1)^2\pi^2}\sin(2n-1)\frac{\pi}{2} \qquad (\because \cos pl = 0)$$

$$= \frac{-8Al}{\pi^2(2n-1)^2}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{8Al}{\pi^2(2n-1)^2}(-1)^n \qquad \ldots(8)$$

$\therefore$ From (7), $$u(x,t) = Ax + \frac{8Al}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n-1)^2}\right]\sin(2n-1)\frac{\pi x}{2l}\, e^{-\left[\frac{(2n-1)^2\pi^2c^2t}{4l^2}\right]}$$

Example 29. *Solve:* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ *under the conditions*

(i) $u \neq \infty$ *if* $t \to \infty$ *(ii)* $\dfrac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$

(iii) $u = lx - x^2$ *for* $t = 0$ *between* $x = 0$ *and* $x = l$.

Sol. Solution to $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ is

$$u(x,t) = c_1 e^{-c^2kt}(c_2\cos cx + c_3 \sin cx) \qquad \ldots(1)$$

Eqn. (1) satisfies the condition $u \neq \infty$ if $t \to \infty$

Applying $\dfrac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (1), we get

$$c_3 = 0$$

and $$c = \frac{n\pi}{l},\ n \in I$$

$\therefore$ $$u = c_1c_2\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)}\cos\frac{n\pi x}{l} = a_n \cos\frac{n\pi x}{l}\, e^{-\left(\frac{n^2\pi^2kt}{l^2}\right)} \qquad \ldots(2)$$

Again, the second possible solution is

$$u = c_1(c_2x + c_3) \qquad \ldots(3) \quad | \text{ if } c^2 = 0$$

Applying $\dfrac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$ to (3), we get $c_2 = 0$

$\therefore$ $$u = c_1c_3 = \frac{a_0}{2} \text{ (say)} \qquad \ldots(4) \quad | \text{ From (3)}$$

$\therefore$ The general solution is the sum of solutions (2) and (4) for various n.

$$\therefore \qquad u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \, e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)} \qquad \ldots(5)$$

Now applying $u = lx - x^2$ for $t = 0$ to eqn. (5), we get

$$lx - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

Here,
$$a_0 = \frac{2}{l} \int_0^l (lx - x^2)\, dx = \frac{l^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l (lx - x^2) \cos \frac{n\pi x}{l}\, dx$$

$$= \begin{cases} -\dfrac{4l^2}{n^2\pi^2}; & \text{when } n \text{ is even} \\ 0; & \text{when } n \text{ is odd} \end{cases} \qquad | \text{ On simplification}$$

$$\therefore \qquad u = \frac{l^2}{6} - \frac{4l^2}{\pi^2} \sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l} \, e^{-\left(\frac{n^2\pi^2 kt}{l^2}\right)}$$

Put $n = 2m$, we get

$$u(x, t) = \frac{l^2}{6} - \frac{l^2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos \frac{2m\pi x}{l} \, e^{-\left(\frac{4m^2\pi^2 kt}{l^2}\right)}$$

EXERCISE D

1. (*i*) Solve: $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$; α constant, subject to the boundary conditions $u(0, t) = 0$, $u(\pi, t) = 0$ and the initial condition $u(x, 0) = \sin 2x$.

(*ii*) Solve: $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ given that

(*a*) $u = 0$ when $x = 0$ and $x = l$ for all t (*b*) $u = 3 \sin \frac{\pi x}{l}$ when $t = 0$ for all x.

(*iii*) Solve: $u_t = a^2 u_{xx}$ under the conditions $u_x(0, t) = 0 = u_x(\pi, t)$ and $u(x, 0) = x^2 (0 < x < \pi)$.

2. (*i*) Determine the solution of one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where the boundary conditions are $u(0, t) = 0$, $u(l, t) = 0$ $(t > 0)$ and the initial condition $u(x, 0) = x$: l being the length of the bar.

(*ii*) Find the temperature distribution in a rod of length 2 m whose end points are fixed at temperature zero and the initial temperature distribution is $f(x) = 100x$.

3. The heat flow in a bar of length 10 cm of homogeneous material is governed by partial diff. eqn. $u_t = c^2 u_{xx}$. The ends of the bar are kept at temp. 0°C and initial temp. is $f(x) = x(10 - x)$.

Find the temp. in the bar at any instant of time.

NOTES

4. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length L cm. with its ends kept at zero temperature and initial temperature given by $\frac{x(L-x)\,d}{L^2}$.

5. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is $u(x, 0) = \begin{cases} x, & 0 \le x \le 50 \\ 100 - x, & 50 \le x \le 100 \end{cases}$

Find the temperature $u(x, t)$ at any time.

6. Find the temperature $u(x, t)$ in a slab whose ends $x = 0$ and $x = L$ are kept at zero temperature and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} k, & \text{when } 0 < x < \frac{1}{2}L \\ 0, & \text{when } \frac{1}{2}L < x < L \end{cases}.$$

7. Find the temperature distribution in a rod of length π which is totally insulated including the ends and the initial temperature distribution is 100 cos x.

8. Find the temperature in a thin metal rod of length L with both ends insulated (so that there is no passage of heat through the ends) and with initial temperature $\sin \frac{\pi x}{L}$ in the rod.

9. (*i*) The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two end are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

(*ii*) A bar 10 cm long, with insulated sides, has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature at A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t.

10. A homogeneous rod of conducting material of length '1' has its ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature distribution.

$$\left[\textbf{Hint: } u(x, 0) = \begin{cases} 2Tx, & 0 \le x \le \frac{1}{2} \\ 2T(1-x), & \frac{1}{2} \le x \le 1 \end{cases} \right]$$

11. Solve $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

(*i*) θ is finite when $t \to \infty$, (*ii*) $\frac{\partial \theta}{\partial x} = 0$ when $x = 0$ and $\theta = 0$ when $x = l$ for all t,

(*iii*) $\theta = \theta_0$ when $t = 0$ for all values of x between 0 and l.

12. Find a solution of the heat conduction equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ such that

(*i*) u is finite when $t \to \infty$, (*ii*) $u = 100$ when $x = 0$ or π for all values of t,

(*iii*) $u = 0$ when $t = 0$ for all values of x between 0 and π.

(Here, the initially ice-cold rod has its ends in boiling water.)

NOTES

13. Solve the following IBVP: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$

$u(0, t) = 1$, $u(1, t) = 2$, $t \geq 0$

$u(x, 0) = 1 + x + 2 \sin \pi x$

14. Solve the following IBVP: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$

$u(0, t) = 1$, $u(1, t) = 3$

$u(x, 0) = 1 + 2x + 3 \sin \pi x$

Answers

1. (*i*) $u(x, t) = \sin 2x\, e^{-4\alpha t}$ (*ii*) $u(x, t) = 3 \sin \dfrac{\pi x}{l} e^{-(a^2\pi^2 t/l^2)}$

(*iii*) $u(x, t) = \dfrac{\pi^3}{3} + 4 \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx\, e^{-a^2 n^2 t}$

2. (*i*) $u(x, t) = -\dfrac{2l}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos n\pi}{n} \sin \dfrac{n\pi x}{l} e^{-\left(\frac{c^2 n^2 \pi^2 t}{l^2}\right)}$

(*ii*) $u(x, t) = -\dfrac{400}{\pi} \sum_{n=1}^{\infty} \dfrac{\cos n\pi}{n} \sin \dfrac{n\pi x}{2} e^{-\left(\frac{c^2 n^2 \pi^2 t}{4}\right)}$

3. $u(x, t) = \dfrac{800}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{10} e^{-\left[\frac{(2n-1)^2 \pi^2 c^2 t}{100}\right]}$.

4. $u(x, t) = \dfrac{8d}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{L} e^{-\frac{(2n-1)^2 \pi^2 c^2 t}{L^2}}$.

5. $u(x, t) = \dfrac{400}{\pi^2} \sum_{m=0}^{\infty} \dfrac{(-1)^m}{(2m+1)^2} \sin \dfrac{(2m+1)\pi x}{100} e^{-\left[\frac{(2m+1) c\pi}{100}\right]^2 t}$

6. $u(x, t) = \dfrac{4k}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin^2 \dfrac{n\pi}{4} \sin \dfrac{n\pi x}{L} e^{-\left(\frac{c^2 n^2 \pi^2 t}{L^2}\right)}$

7. $100\, e^{-t} \cos x$

8. $u(x, t) = \dfrac{2}{\pi} - \dfrac{4}{\pi} \sum_{m=1}^{\infty} \dfrac{1}{(4m^2 - 1)} \cos \left(\dfrac{2m\pi x}{L}\right) e^{-\frac{4m^2 \pi^2 c^2 t}{L^2}}$

9. (*i*) $u(x, t) = \dfrac{200}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin \dfrac{n\pi x}{50} e^{-\left(\frac{n^2 \pi^2 kt}{2500}\right)}$

(*ii*) $u(x, t) = 90 - 3x - \dfrac{80}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin \dfrac{n\pi x}{5} e^{-\left(\frac{c^2 n^2 \pi^2 t}{25}\right)}$

10. $u(x, t) = \dfrac{8T}{\pi^2} \sum_{m=1}^{\infty} \dfrac{(-1)^{m+1}}{(2m-1)^2} \sin (2m-1)\pi x\, e^{-[(2m-1)^2 \pi^2 c^2 t]}$

11. $\theta = \dfrac{4\theta_0}{\pi} \left[e^{-(\pi/2l)^2 kt} \cos \dfrac{\pi x}{2l} - \dfrac{1}{3} e^{-\left(\frac{3\pi}{2l}\right)^2 kt} \cos \dfrac{3\pi x}{2l} + \dfrac{1}{5} e^{-\left(\frac{5\pi}{2l}\right)^2 kt} \cos \dfrac{5\pi x}{2l} - \ldots \right]$

12. $u(x, t) = 100 - \dfrac{400}{\pi} \sum_{m=1}^{\infty} \dfrac{\sin (2m-1)x}{2m-1} e^{-(2m-1)^2 \alpha t}$

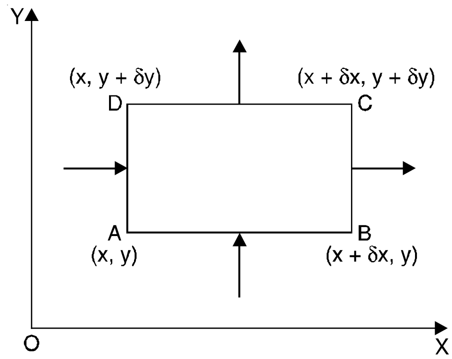
13. $u(x, t) = 1 + x + 2e^{-\pi^2 t} \sin \pi x$

14. $u(x, t) = 1 + 2x + 3e^{-\pi^2 t} \sin \pi x.$

TWO-DIMENSIONAL HEAT FLOW

NOTES

Consider the flow of heat in a metal plate, in the XOY plane. If the temperature at any point is independent of the z-coordinate and depends on x, y and t only, then the flow is called two dimensional and the heat-flow lies in the plane XOY only and is zero along the normal to the plane XOY.



Take a rectangular element of the plate with sides δx and δy and thickness α. As discussed in the one-dimensional heat flow along a bar, the quantity of heat that enters the plate per second from the sides AB and AD is given by

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_y \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_x$$

respectively and that which flows out through the sides CD and BC per second is

$$-k\alpha\,\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} \quad \text{and} \quad -k\alpha\,\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ respectively.}$$

Therefore, the total gain of heat by the rectangular plate ABCD per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad \ldots(1)$$

The rate of gain of heat by the plate is also given by

$$s\rho\delta x\delta y\,\frac{\partial u}{\partial t} \quad \ldots(2)$$

where s = specific heat and ρ = density of the metal plate.

Equating (1) and (2), we obtain

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = s\rho\delta x\delta y\,\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking the limit as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = s\rho\,\frac{\partial u}{\partial t}$$

or

$$\boxed{c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}} \quad \text{where } c^2 = \frac{k}{s\rho} \quad \ldots(3)$$

Equation (3) gives the temperature distribution of the plate in the transient state.

Note 1. In steady state, u is independent of t, so that $\frac{\partial u}{\partial t} = 0$ and the above equation reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(4)$$

which is known as **Laplace's Equation in two dimensions.**

Note 2. The equation of heat flow in a solid (Three-dimensional heat flow) can similarly be derived as

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}$$

In the steady state, it reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

which is **Laplace's Equation in three dimensions.**

SOLUTION OF LAPLACE'S EQUATION IN TWO DIMENSIONS

Laplace's equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Let $u = XY$ ...(2)

where X is a function of x only and Y is a function of y only, be a solution of (1).

Then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$

Substituting in (1), we have $X''Y + XY'' = 0$ or $\frac{X''}{X} = -\frac{Y''}{Y}$ ...(3)

Now the LHS of (3) is a function of x only and the RHS is a function of y only. Since x and y are independent variables, this equation can hold only when both sides reduce to a constant, say k. Then equation (3) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + kY = 0 \qquad ...(4)$$

Solving equations (4), we get

(*i*) *When k is positive* and $= p^2$, say

$$X = c_1 e^{px} + c_2 e^{-px}, \ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative* and $= -p^2$, say

$$X = c_1 \cos px + c_2 \sin px, \ Y = c_3 e^{py} + c_4 e^{-py}$$

(*iii*) *When k = 0*

$$X = c_1 x + c_2, \ Y = c_3 y + c_4$$

Thus, the various possible solutions of Laplace's equation (1) are:

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(5)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(6)$$

$$u = (c_1 x + c_2)(c_3 y + c_4) \qquad ...(7)$$

NOTES

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions. Solution (6) is the required solution.

$$u(x, y) = (c_1 \cos px + c_2 \sin px)\,(c_3\, e^{py} + c_4 e^{-py}).$$

SOLVED EXAMPLES

Example 30. *Use separation of variables method to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin \frac{n\pi x}{l}$.

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

Let $\qquad u = XY \qquad$ ...(2)

where X is a function of x only and Y is a function of y only then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = X\frac{d^2Y}{dy^2}$$

$\therefore$ From (1), $\qquad YX'' + XY'' = 0$

$\Rightarrow \qquad \dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$

Case I. When $\qquad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = p^2$ (say)

(*i*) $\qquad \dfrac{X''}{X} = p^2$

$X'' - p^2X = 0$

Auxiliary equation is $\quad m^2 - p^2 = 0$

$m = \pm p$

$\therefore \qquad$ C.F. $= c_1e^{px} + c_2e^{-px}$

P.I. $= 0$

$\therefore \qquad X = c_1e^{px} + c_2e^{-px}$

(*ii*) $\qquad \dfrac{-Y''}{Y} = p^2 \quad \Rightarrow \quad Y'' + p^2Y = 0$

Auxiliary equation is $\quad m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$\therefore \qquad$ C.F. $= c_3 \cos py + c_4 \sin py$

P.I. $= 0$

$\therefore \qquad y = c_3 \cos py + c_4 \sin py$

Now, $\qquad X(0) = 0$

$\Rightarrow \qquad c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$

$X(l) = 0$

$\Rightarrow \qquad c_1e^{pl} + c_2e^{-pl} = 0 \quad \Rightarrow \quad c_1\,(e^{pl} - e^{-pl}) = 0$

$\Rightarrow \quad c_1 = 0 \qquad | \text{ Since } e^{pl} - e^{-pl} \neq 0 \text{ (as } p \neq 0 \neq l)$

$\therefore \quad c_2 = 0$

$\therefore \quad X = 0 \quad \Rightarrow \quad u = XY = 0$ which is impossible

Hence we reject case I.

Case II. When $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = 0$ (say)

(*i*) $\dfrac{X''}{X} = 0$

$\Rightarrow \quad X'' = 0 \quad \Rightarrow \quad X = c_5 x + c_6$

(*ii*) $\dfrac{-Y''}{Y} = 0$

$\Rightarrow \quad Y'' = 0 \quad \Rightarrow \quad Y = c_7 y + c_8$

Now, $\quad X(0) = 0 \quad \Rightarrow \quad c_6 = 0$

$X(l) = 0$

$\Rightarrow \quad c_5 l + c_6 = 0 \quad \Rightarrow c_5 l = 0$

$\Rightarrow \quad c_5 = 0 \qquad (\text{Since } l \neq 0)$

$\therefore \quad X = 0$

$\therefore \quad u = XY = 0$ which is impossible

Hence we also reject case II.

Case III. When $\dfrac{X''}{X} = -\dfrac{Y''}{Y} = -p^2$ (say)

(*i*) $\dfrac{X''}{X} = -p^2$

$\Rightarrow \quad X'' + p^2 X = 0 \quad \Rightarrow \quad \dfrac{d^2X}{dx^2} + p^2 X = 0.$

Auxiliary equation is $\quad m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$\text{C.F.} = c_9 \cos px + c_{10} \sin px$

$\text{P.I.} = 0$

$X = c_9 \cos px + c_{10} \sin px$

(*ii*) $-\dfrac{Y''}{Y} = -p^2$

$\Rightarrow \quad \dfrac{Y''}{Y} = p^2 \quad \Rightarrow \quad \dfrac{d^2Y}{dy^2} - p^2 Y = 0.$

Auxiliary equation is

$m^2 - p^2 = 0$

$m = \pm p.$

$\therefore \quad \text{C.F.} = c_{11} e^{py} + c_{12} e^{-py}$

$\text{P.I.} = 0$

Hence, $\quad Y = c_{11} e^{py} + c_{12} e^{-py}.$

Now, $\quad X(0) = 0 \quad \Rightarrow \quad c_9 = 0$

$\therefore \quad X = c_{10} \sin px$

$X(l) = 0$

$c_{10} \sin pl = 0$

NOTES

$\Rightarrow \quad \sin pl = 0 = \sin n\pi,\ n \in I$

$\therefore \quad p = \dfrac{n\pi}{l}$

$\therefore \quad X = c_{10} \sin \dfrac{n\pi x}{l}$ ...(3)

Again, $\quad Y(0) = 0$

$\Rightarrow \quad c_{11} + c_{12} = 0 \Rightarrow c_{12} = -c_{11}$

$$Y = c_{11}(e^{py} - e^{-py}) = c_{11}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) \quad ...(4)$$

$\therefore \quad u = XY = c_{10}c_{11} \sin \dfrac{n\pi x}{l}\ [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$

or $\quad u(x, y) = b_n \sin \dfrac{n\pi x}{l}\ [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$ ...(5)

Now, $\quad u(x, a) = \sin \dfrac{n\pi x}{l} = b_n \sin \dfrac{n\pi x}{l}\ [e^{(n\pi a/l)} - e^{-(n\pi a/l)}]$

$$\Rightarrow \quad b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}} = \frac{1}{2 \sin h\left(\frac{n\pi a}{l}\right)}$$

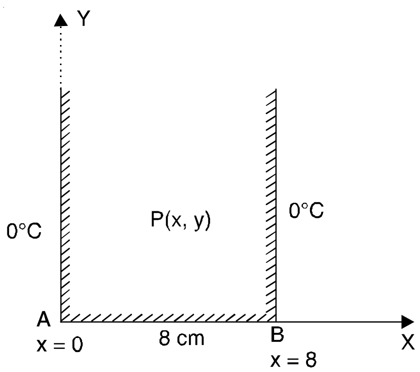
$$\therefore \quad u(x, y) = \frac{e^{(n\pi y/l)} - e^{-(n\pi y/l)}}{2 \sinh\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l} = \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l}.$$

Example 31. *A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by*

$$u(x, 0) = 100 \sin \frac{\pi x}{8},\ 0 < x < 8$$

while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0°C, show that the steady state temperature at any point of the plate is given by

$$u(x, y) = 100 e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}.$$



Sol. Let $u(x, y)$ be the temperature at any point P of the plate.

Two dimensional heat flow equation in steady state is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

Its solution is $\quad u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ ...(2)

Boundary conditions are

$$u(0, y) = 0 = u(8, y)$$

$$\underset{y \to \infty}{\text{Lt}}\ u(x, y) = 0$$

NOTES

$$u(x, 0) = 100 \sin \frac{\pi x}{8}, \quad 0 < x < 8$$

From (2), $\quad u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad c_1 = 0.$

$\therefore$ From (2), $\quad u(x, y) = c_2 \sin px\, (c_3 e^{py} + c_4 e^{-py}) \quad \ldots(3)$

$u(8, y) = 0 = c_2 \sin 8p(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad \sin 8p = 0 = \sin n\pi$

$\Rightarrow \quad p = \frac{n\pi}{8} \quad (n \in \mathrm{I})$

$\therefore$ From (3), $\quad u(x, y) = c_2 \sin \frac{n\pi x}{8} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}}) \quad \ldots(4)$

$$\underset{y \to \infty}{\text{Lt}}\ u(x, y) = 0 = c_2 \sin \frac{n\pi x}{8} \lim_{y \to \infty} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}})$$

which is satisfied only when

$$c_3 = 0.$$

$\therefore$ From (4), $\quad u(x, y) = c_2 c_4 \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} \quad \ldots(5)$

From (5), $\quad u(x, 0) = 100 \sin \frac{\pi x}{8} = b_n \sin \frac{n\pi x}{8}$

$\Rightarrow \quad b_n = 100,\ n = 1.$

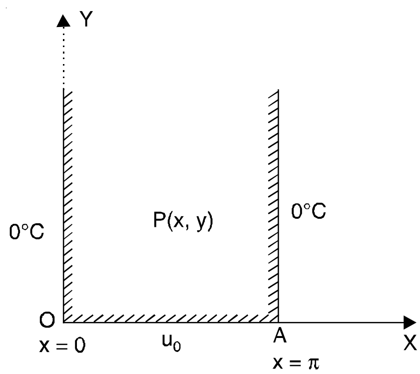
$\therefore$ From (5), $\quad u(x, y) = 100 \sin \left(\frac{\pi x}{8}\right) e^{-(\pi y/8)}$

which is the required steady state temperature at any point of the plate.

Example 32. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.*

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(1)$$



Boundary conditions are,

$$u(0, y) = 0 = u(\pi, y)$$

$$\underset{y \to \infty}{\text{Lt}}\ u(x, y) = 0 \ (0 < x < \pi)$$

and $\quad u(x, 0) = u_0 \ (0 < x < \pi)$

Solution to equation (1) is

$u(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \ldots(2)$

From (2), $\quad u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow \quad c_1 = 0.$

From (2), $\quad u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$$\Rightarrow \qquad \sin p\pi = 0 = \sin n\pi \ (n \in I) \qquad ...(3)$$

$$\therefore \qquad p = n.$$

$$\therefore \text{ From (3),} \qquad u(x, y) = c_2 \sin nx\,(c_3 e^{ny} + c_4 e^{-ny}) \qquad ...(4)$$

$$\lim_{y \to \infty} u(x, y) = 0 = c_2 \sin nx \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

which is satisfied only when $c_3 = 0$.

$\therefore$ From (4), $u(x, y) = c_2 c_4 e^{-ny} \sin nx = b_n e^{-ny} \sin nx$, where $c_2 c_4 = b_n$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx \qquad ...(5)$$

$$u(x, 0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin nx \, dx$$

$$= \frac{2u_0}{\pi} \left(\frac{-\cos nx}{n} \right)_0^{\pi} = \frac{2u_0}{n\pi} \{1 - (-1)^n\}$$

$$= \begin{cases} \dfrac{4u_0}{n\pi}; & \text{if } n \text{ is odd} \\ 0; & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \text{ From (5),} \qquad u(x, y) = \frac{4u_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n} e^{-ny} \qquad (n \text{ is odd})$$

or

$$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1)\, x\, e^{-(2n-1)y}.$$

Example 33. *A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by*

and

$$u(x, y) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$$

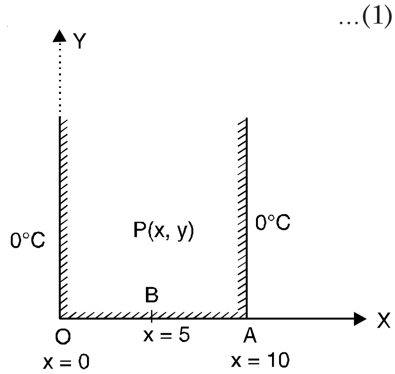
and the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y).

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(2)$$



Boundary conditions are $u(0, y) = 0$

$$u(10, y) = 0$$

$$\lim_{y \to \infty} u(x, y) = u(x, \infty) = 0$$

and

$$u(x, 0) = \begin{cases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x \le 10 \end{cases}$$

From (2), $u(x, y) = 0 = c_1(c_3e^{py} + c_4e^{-py})$

$\Rightarrow$ $c_1 = 0$

From (2), $u(x, y) = c_2 \sin px \,(c_3 e^{py} + c_4 e^{-py})$ ...(3)

$u(10, y) = 0 = c_2 \sin 10p\, (c_3e^{py} + c_4e^{-py})$

$\Rightarrow$ $\sin 10p = 0 = \sin n\pi$

or $10p = n\pi \; (n \in \text{I})$

$\Rightarrow$ $p = \dfrac{n\pi}{10}.$

$\therefore$ From (3), $u(x, y) = c_2 \sin \dfrac{n\pi x}{10} (c_3 e^{\frac{n\pi y}{10}} + c_4 e^{-\frac{n\pi y}{10}})$ ...(4)

$$\lim_{y \to \infty} u(x, y) = c_2 \sin \frac{n\pi x}{10} \lim_{y \to \infty} (c_3 e^{\frac{n\pi y}{10}} + c_4 e^{-\frac{n\pi y}{10}})$$

which is satisfied only when $c_3 = 0$.

Hence from (4), $u(x, y) = c_2c_4 \sin \dfrac{n\pi x}{10} e^{-\frac{n\pi y}{10}} = b_n \sin \dfrac{n\pi x}{10} e^{-\frac{n\pi y}{10}}$ ...(5)

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \quad \text{...(6)}$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10},$$

where $b_n = \dfrac{2}{10} \displaystyle\int_0^{10} u(x, 0) \sin \dfrac{n\pi x}{10}\, dx$

$$= \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10} dx + \int_5^{10} 20\,(10 - x) \sin \frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_0^5 - \int_0^5 1\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)dx + \left\{(10 - x)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_5^{10} - \int_5^{10} (-1)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)dx\right]$$

$$= 4\left[\frac{10}{n\pi}(-5)\cos\frac{n\pi}{2} + \frac{10}{n\pi}\left(\frac{\sin \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_0^5 + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{10}{n\pi}\left(\frac{\sin \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_5^{10}\right]$$

$$= 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

From (6), $u(x, y) = \dfrac{800}{\pi^2} \displaystyle\sum_1^{\infty} \dfrac{\sin n\pi/2}{n^2} \sin \dfrac{n\pi x}{10} e^{-\frac{n\pi y}{10}}.$

NOTES

Example 34. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ *= 0, 0 < x < π, 0 < y < π, which satisfies the conditions :*

$$u(0, y) = u(\pi, y) = u(x, \pi) = 0 \text{ and } u(x, 0) = \sin^2 x.$$

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Its solution consistent with boundary conditions is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \quad \text{...(2)}$$

From (2), $u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (2), $u(x, y) = c_2 \sin px\ (c_3 e^{py} + c_4 e^{-py})$ ...(3)

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi\ (n \in I)$

$\therefore$ $p = n.$

Hence from (3), $u(x, y) = c_2 \sin nx\ (c_3 e^{ny} + c_4 e^{-ny}) = \sin nx\ (Ae^{ny} + Be^{-ny})$ ...(4)

where $c_2 c_3 = A$ and $c_2 c_4 = B.$

From (4), $u(x, \pi) = \sin nx(Ae^{n\pi} + Be^{-n\pi})$

$0 = \sin nx(Ae^{n\pi} + Be^{-n\pi})$

$\Rightarrow$ $0 = Ae^{n\pi} + Be^{-n\pi}$

$\Rightarrow$ $Ae^{n\pi} = -Be^{-n\pi} = -\frac{1}{2} B_n$ (say)

then (4) becomes, $u(x, y) = \sin nx \left[-\frac{1}{2} B_n e^{-n\pi} e^{ny} + \frac{1}{2} B_n e^{n\pi} e^{-ny}\right]$

$$= \frac{1}{2} B_n[e^{n(\pi-y)} - e^{-n(\pi-y)}] \sin nx = B_n \sin hn\ (\pi - y) \sin nx.$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin hn\ (\pi - y) \sin nx \quad \text{...(5)}$$

$$u(x, 0) = \sin^2 x = \sum_{n=1}^{\infty} B_n \sin hn\pi \sin nx$$

where

$$B_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} \sin^2 x \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (1 - \cos 2x) \sin nx\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \left[\sin nx - \frac{1}{2}\{\sin (n+2) x + \sin (n-2) x\}\right] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos nx}{n} + \frac{\cos (n+2)x}{2(n+2)} + \frac{\cos (n-2)x}{2(n-2)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left[\left(\frac{1}{n+2} + \frac{1}{n-2} - \frac{2}{n}\right)\{(-1)^n - 1\}\right], \text{ when } n \neq 2$$

NOTES

$$B_n \sinh n\pi = \begin{cases} \dfrac{-8}{\pi n(n^2-4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even and} \neq 2 \end{cases}$$

when $n = 2$,

$$B_2 \sinh 2\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin 2x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(1-\cos 2x)\sin 2x\, dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin 2x - \frac{1}{2}\sin 4x\right)dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2} + \frac{1}{8}\cos 4x\right)_0^{\pi} = 0$$

$\therefore \qquad B_2 = 0.$

Hence the solution (5) becomes,

$$u(x, y) = \frac{-8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx \sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$$

or

$$u(x, y) = -\frac{8}{\pi}\sum_{m=1,2,3,\ldots}^{\infty} \frac{\sin (2m-1)x \sinh (2m-1)(\pi - y)}{(2m-1)\{(2m-1)^2-4\}\sinh (2m-1)\pi}.$$

Example 35. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$*, with the rectangle* $0 \le x \le a$*,* $0 \le y \le b$ *; given that*

$$u(x, b) = u(0, y) = u(a, y) = 0 \text{ and } \quad u(x, 0) = x(a - x).$$

Sol. The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(2)$$

$$u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (2), $\quad u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(3)$

$$u(a, y) = 0 = c_2 \sin ap\,(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad \sin ap = 0 = \sin n\pi \ (n \in I)$

$\Rightarrow \qquad ap = n\pi \quad \text{or} \quad p = \dfrac{n\pi}{a}.$

$\therefore$ From (3), $\quad u(x, y) = c_2 \sin \dfrac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

$$u(x, y) = \sin \frac{n\pi x}{a}\left(A\, e^{\frac{n\pi y}{a}} + B\, e^{-\frac{n\pi y}{a}}\right) \qquad \ldots(4)$$

where $\quad c_2 c_3 = A \quad$ and $\quad c_2 c_4 = B$

$$u(x, b) = \sin \frac{n\pi x}{a}\left(A\, e^{\frac{n\pi b}{a}} + B\, e^{-\frac{n\pi b}{a}}\right)$$

$$0 = \sin \frac{n\pi x}{a}\left(A\, e^{\frac{n\pi b}{a}} + B\, e^{-\frac{n\pi b}{a}}\right)$$

NOTES

$$\Rightarrow \quad A\,e^{\frac{n\pi b}{a}} + Be^{-\frac{n\pi b}{a}} = 0$$

$$A\,e^{\frac{n\pi b}{a}} = -Be^{-\frac{n\pi b}{a}} = -\frac{1}{2}B_n \text{ (say).}$$

Then (4) becomes,

$$u(x, y) = \sin\frac{n\pi x}{a}\left[\frac{-1}{2}B_n e^{-\frac{n\pi b}{a}} e^{\frac{n\pi y}{a}} + \frac{1}{2}B_n e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right]$$

$$= \frac{1}{2}B_n \sin\frac{n\pi x}{a}\,[e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}]$$

$$= \frac{1}{2}B_n \sin\frac{n\pi x}{a}\,.\,2\sinh\frac{n\pi}{a}(b-y) = B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(b-y).$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(b-y) \qquad ...(5)$$

Applying to this the condition $u(x, 0) = x(a-x)$, we get

From (5), $$u(x, 0) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}$$

$$\Rightarrow \quad x(a-x) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}$$

where $$B_n \sinh\frac{n\pi}{a}b = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi}{a}x\,dx$$

$$= \frac{2}{a}\left[\left\{(ax-x^2)\left(\frac{-\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right)\right\}_0^a - \int_0^a (a-2x)\,.\left(\frac{-\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right)dx\right]$$

$$= \frac{2}{a}\,.\,\frac{a}{n\pi}\int_0^a (a-2x)\,.\cos\frac{n\pi}{a}x\,dx$$

$$= \frac{2}{n\pi}\left[\left\{(a-2x)\frac{\sin\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right\}_0^a - \int_0^a (-2)\left(\frac{\sin\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right)dx\right]$$

$$= \frac{4}{n\pi}\,.\,\frac{a}{n\pi}\left(\frac{-\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right)_0^a = \frac{4a}{n^2\pi^2}\,.\,\frac{a}{n\pi}(1-\cos n\pi)$$

$$= \frac{4a^2}{n^3\pi^3}[1-(-1)^n] = \begin{cases}\dfrac{8a^2}{n^3\pi^3}, & \text{when } n \text{ is odd}\\ 0, & \text{when } n \text{ is even}\end{cases}$$

$$\therefore \quad B_n = \begin{cases}\dfrac{8a^2}{\sinh\left(\frac{n\pi}{a}b\right)(n^3\pi^3)}, & \text{when } n \text{ is odd}\\ 0, & \text{when } n \text{ is even}\end{cases}$$

$$\therefore \quad \text{From (5)}, \; u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{\sin \frac{n\pi x}{a}}{n^3 \sinh \frac{n\pi}{a} b} \cdot \sinh \frac{n\pi}{a}(b - y)$$

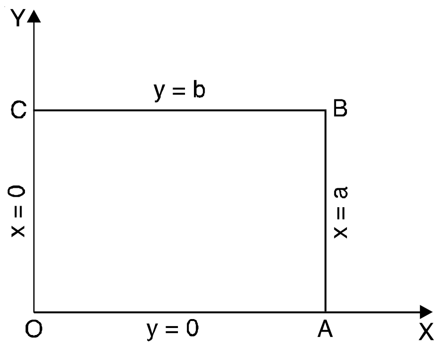
(n is odd)

or
$$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin (2n+1) \frac{\pi x}{a} \cdot \frac{\sinh \frac{(2n+1)\pi}{a}(b-y)}{\sinh \frac{(2n+1)\pi}{a} b}$$

Example 36. *A thin rectangular plate whose surface is impervious to heat flow has at t = 0 an arbitrary distribution of temperature f(x, y). Its four edges x = 0, x = a, y = 0, y = b are kept at zero temperature. Determine the temperature at a point of a plate as t increases. Discuss the problem when* $f(x, y) = \beta \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$.

Sol. Two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}. \quad \ldots(1)$$



Boundary conditions are

$$u(0, y, t) = 0 = u(a, y, t)$$
$$u(x, 0, t) = 0 = u(x, b, t)$$

and $\quad u(x, y, t) = f(x, y)$ at $t = 0$.

Let the solution be $\quad u = \text{XYT}$

where X is a function of x only, Y is a function of y only and T is a function of t only.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\text{XYT}) = \text{XY} \frac{d\text{T}}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(\text{XYT}) = \text{YT} \frac{d^2\text{X}}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(\text{XYT}) = \text{XT} \frac{d^2\text{Y}}{dy^2}.$$

From (1), $\text{YT X}'' + \text{XTY}'' = \frac{1}{c^2}(\text{XYT}')$

$$\Rightarrow \quad \frac{\text{X}''}{\text{X}} + \frac{\text{Y}''}{\text{Y}} = \frac{\text{T}'}{c^2\text{T}} \quad \ldots(2)$$

There are three possibilities:

(*i*) $\quad \frac{\text{X}''}{\text{X}} = 0, \quad \frac{\text{Y}''}{\text{Y}} = 0, \quad \frac{\text{T}'}{c^2\text{T}} = 0$

(*ii*) $\quad \frac{\text{X}''}{\text{X}} = \text{K}_1^2, \quad \frac{\text{Y}''}{\text{Y}} = \text{K}_2^2, \quad \frac{\text{T}'}{c^2\text{T}} = \text{K}^2$

(*iii*) $\quad \frac{\text{X}''}{\text{X}} = -\text{K}_1^2, \quad \frac{\text{Y}''}{\text{Y}} = -\text{K}_2^2, \quad \frac{\text{T}'}{c^2\text{T}} = -\text{K}^2$

where $\quad \text{K}^2 = \text{K}_1^2 + \text{K}_2^2$.

Of these three solutions, we have to select the solution which is consistent with the physical nature of the problem.

The solution satisfying the given boundary conditions will be given by (*iii*).

Then, $X = c_1 \cos K_1 x + c_2 \sin K_1 x$

$Y = c_3 \cos K_2 y + c_4 \sin K_2 y$

$T = c_5 e^{-c^2 K^2 t}$

$\therefore \quad u = XYT$

$$\Rightarrow \quad u(x, y, t) = (c_1 \cos K_1 x + c_2 \sin K_1 x)(c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{-c^2 K^2 t}) \quad ...(3)$$

$$u(0, y, t) = 0 = c_1 (c_3 \cos K_2 y + c_4 \sin K_2 y) c_5 e^{-c^2 K^2 t}$$

$\Rightarrow \quad c_1 = 0.$

$\therefore$ From (3), $u(x, y, t) = c_2 \sin K_1 x (c_3 \cos K_2 y + c_4 \sin K_2 y)(c_5 e^{-c^2 K^2 t})$

$$= c_6 \sin K_1 x \,(c_3 \cos K_2 y + c_4 \sin K_2 y)(e^{-c^2 K^2 t}) \quad ...(4)$$

where $c_2 c_5 = c_6$

From (4), $u(a, y, t) = 0 = c_6 \sin K_1 a (c_3 \cos K_2 y + c_4 \sin K_2 y) e^{-c^2 K^2 t}$

$\Rightarrow \quad \sin K_1 a = 0 = \sin n\pi \; (n \in I)$

$\therefore \quad K_1 = \dfrac{n\pi}{a}.$

From (4), $$u(x, y, t) = c_6 \sin \frac{n\pi x}{a} (c_3 \cos K_2 y + c_4 \sin K_2 y)(e^{-c^2 K^2 t}) \quad ...(5)$$

$$u(x, 0, t) = 0 = c_6 \sin \frac{n\pi x}{a} . c_3 e^{-c^2 K^2 t}$$

$\Rightarrow \quad c_3 = 0.$

$\therefore$ From (5), $$u(x, y, t) = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 y \, e^{-c^2 K^2 t} \quad ...(6)$$

$$u(x, b, t) = 0 = c_6 c_4 \sin \frac{n\pi x}{a} \sin K_2 b \, e^{-c^2 K^2 t}$$

$\Rightarrow \quad \sin K_2 b = 0 = \sin m\pi \; (m \in I)$

$K_2 b = m\pi$

$\Rightarrow \quad K_2 = \dfrac{m\pi}{b}.$

$\therefore$ From (6), $$u(x, y, t) = c_6 c_4 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 K^2 t}$$

$$= A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 K^2 t} \quad ...(7)$$

| where $c_6 c_4 = A_{mn}$

But, $$K^2 = K_1^2 + K_2^2 = \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}$$

or $$K^2_{mn} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right).$$

By using K_{mn}, equation (7) becomes,

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 K^2_{mn} t} \quad \ldots(8)$$

is the most general solution.

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

which is the double Fourier half-range sine series for $f(x, y)$.

where $$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} f(x, y) dx\, dy.$$

When $f(x, y) = \beta \sin \left(\frac{\pi x}{a}\right) \sin \left(\frac{\pi y}{b}\right)$

$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^{a} \int_{y=0}^{b} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \beta \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dxdy$$

$$= \frac{4\beta}{ab} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{\pi x}{a} dx \int_0^b \sin \frac{m\pi y}{b} \sin \frac{\pi y}{b} dy$$

$$= \frac{\beta}{ab} \int_0^a \left[\cos (n-1) \frac{\pi x}{a} - \cos (n+1) \frac{\pi x}{a}\right] dx$$

$$\times \int_0^b \left[\cos (m-1) \frac{\pi y}{b} - \cos (m+1) \frac{\pi y}{b}\right] dy$$

$$= \frac{\beta}{ab} \left[\frac{\sin (n-1) \frac{\pi x}{a}}{(n-1) \frac{\pi}{a}} - \frac{\sin (n+1) \frac{\pi x}{a}}{(n+1) \frac{\pi}{a}}\right]_0^a$$

$$\times \left[\frac{\sin (m-1) \frac{\pi y}{b}}{(m-1) \frac{\pi}{b}} - \frac{\sin (m+1) \frac{\pi y}{b}}{(m+1) \frac{\pi}{b}}\right]_0^b$$

$$= \frac{\beta}{ab} \cdot 0 = 0 \qquad (\because \sin n\pi = 0 \text{ and } \sin m\pi = 0)$$

Hence from (8), $y(x, y, t) = 0$ when $f(x, y) = \beta \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$.

Example 37. *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *in a rectangle in the xy-plane with $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ and $u(a, y) = f(y)$ parallel to y-axis.*

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(1)$$

Let $$u = XY \quad ...(2)$$
where X is a function of x only and Y is a function of y only. Then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = YX''$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = XY''$$

$\therefore$ From (1), $YX'' + XY'' = 0 \Rightarrow \dfrac{Y''}{Y} = -\dfrac{X''}{X}$ ...(3)

Case I. *When* $\dfrac{Y''}{Y} = -\dfrac{X''}{X} = p^2$ *(say)*

(*i*) $$\frac{Y''}{Y} = p^2$$

$\Rightarrow$ $$Y'' - p^2Y = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ $$\text{C.F.} = c_1e^{py} + c_2e^{-py}$$

$$\text{P.I.} = 0$$

$\therefore$ $$Y = c_1e^{py} + c_2e^{-py}$$

(*ii*) $$-\frac{X''}{X} = p^2$$

$\Rightarrow$ $$X'' + p^2X = 0$$

Auxiliary equation is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_3 \cos px + c_4 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_3 \cos px + c_4 \sin px$$

Now, $$Y(0) = 0$$

$\Rightarrow$ $$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

$$Y(b) = 0$$

$\Rightarrow$ $$c_1e^{pb} + c_2e^{-pb} = 0$$

$\Rightarrow$ $$c_1(e^{pb} - e^{-pb}) = 0$$

$\Rightarrow$ $$c_1 = 0$$ | Since $e^{pb} - e^{-pb} \neq 0$ (as $p \neq 0 \neq b$)

$\therefore$ $$Y = 0 \Rightarrow u = XY = 0$$ which is impossible.

Hence, we reject case I.

Case II. *When* $\dfrac{Y''}{Y} = -\dfrac{X''}{X} = 0$ *(say)*

(*i*) $$\frac{Y''}{Y} = 0$$

$\Rightarrow$ $$Y'' = 0 \Rightarrow Y = c_5 + c_6y$$

(*ii*) $$-\frac{X''}{X} = 0$$

$\Rightarrow$ $$X'' = 0 \Rightarrow X = c_7 + c_8x$$

Now, $Y(0) = 0 \Rightarrow c_5 = 0$

$Y(b) = 0 \Rightarrow c_6 b = 0 \Rightarrow c_6 = 0$ | $\because b \neq 0$

$\therefore Y = 0$

$\therefore u = XY = 0$ which is impossible

Hence, we also reject case II.

Case III. *When* $\dfrac{Y''}{Y} = -\dfrac{X''}{X} = -p^2$ *(say)*

(*i*) $$\frac{Y''}{Y} = -p^2$$

$\Rightarrow Y'' + p^2 Y = 0$

Auxiliary equation is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$\therefore$ C.F. $= c_9 \cos py + c_{10} \sin py$

P.I. $= 0$

$\therefore Y = c_9 \cos py + c_{10} \sin py$

(*ii*) $$-\frac{X''}{X} = -p^2 \Rightarrow X'' - p^2 X = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ C.F. $= c_{11} e^{px} + c_{12} e^{-px}$

P.I. $= 0$

$\therefore X = c_{11} e^{px} + c_{12} e^{-px}$

Now, $Y(0) = 0 \Rightarrow c_9 = 0$

$Y(b) = 0 \Rightarrow c_{10} \sin bp = 0$

$\therefore \sin bp = 0 = \sin n\pi, n \in I$

$$p = \frac{n\pi}{b}$$

Hence, $$u = XY = c_{10} \sin \frac{n\pi y}{b} \left(c_{11} e^{\frac{n\pi x}{b}} + c_{12} e^{-\frac{n\pi x}{b}} \right) \quad ...(4)$$

Now, $$u(0, y) = 0 = c_{10} \sin \frac{n\pi y}{b} (c_{11} + c_{12})$$

$\Rightarrow c_{11} + c_{12} = 0 \Rightarrow c_{12} = -c_{11}$

$\therefore$ From (4), $$u(x, y) = c_{10} c_{11} \sin \frac{n\pi y}{b} \left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}} \right)$$

$$= b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \quad ...(5)$$

| where $b_n = 2 c_{10} c_{11}$

Most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \quad ...(6)$$

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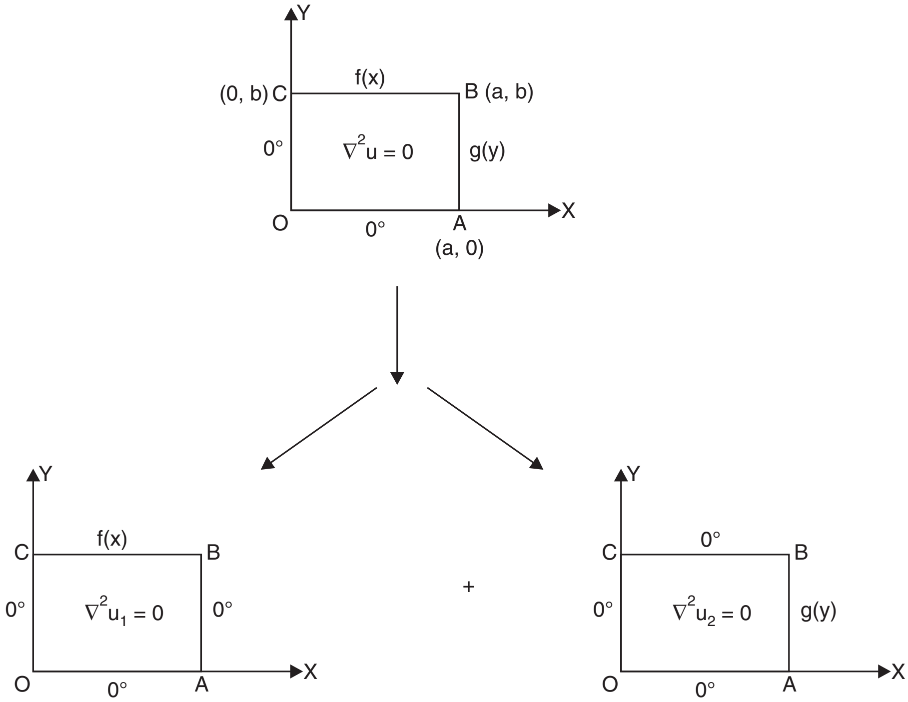
Now, $$u(a, y) = f(y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b}$$

where $$\left(\sinh \frac{n\pi a}{b}\right) b_n = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy$$

$\Rightarrow$ $$b_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy. \quad ...(7)$$

Example 38. *Find the steady state temperature distribution in a rectangular thin plate with its two surfaces insulated and with the conditions u(0, y) = 0, u(x, 0) = 0, u(a, y) = g(y), u(x, b) = f(x).*

Sol. Superposition applied to boundary conditions dismantles the given problem to solution of two simpler problems each of which can easily be solved by the method of separation of variables.



Now, the following two problems are required to be solved:

Problem 1. $\dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0$ subject to conditions

$$u_1(0, y) = 0,\ u_1(a, y) = 0,\ u_1(x, 0) = 0,\ u_1(x, b) = f(x)$$

Problem 2. $\dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\partial^2 u_2}{\partial y^2} = 0$ subject to conditions

$$u_2(0, y) = 0,\ u_2(x, 0) = 0,\ u_2(x, b) = 0,\ u_2(a, y) = g(y)$$

Let us now proceed to solve problem 1.

$$u_1(x, y) = (c_1 \cos px + c_2 \sin px)\ (c_3 e^{py} + c_4\ e^{-py}) \quad ...(1)$$

$$u_1(0, y) = 0 = c_1\,(c_3e^{py} + c_4e^{-py})$$

$\Rightarrow \quad c_1 = 0$

$\therefore \quad u_1(x, y) = c_2 \sin px\,(c_3e^{py} + c_4e^{-py}) \quad \ldots(2)$

$$u_1(a, y) = 0 = c_2 \sin pa\,(c_3e^{py} + c_4e^{-py})$$

$\Rightarrow \quad \sin ap = 0 = \sin n\pi \quad (n \in I)$

$$p = \frac{n\pi}{a}$$

Hence, from (2), $u_1(x, y) = c_2 \sin \frac{n\pi x}{a}\left(c_3\, e^{\frac{n\pi y}{a}} + c_4\, e^{-\frac{n\pi y}{a}}\right) \quad \ldots(3)$

$$u_1(x, 0) = 0 = c_2 \sin \frac{n\pi x}{a}\,(c_3 + c_4)$$

$\Rightarrow \quad c_3 + c_4 = 0 \;\Rightarrow\; c_4 = -c_3$

$\therefore \quad u_1(x, y) = c_2c_3 \sin \frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) = 2\,c_2c_3 \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \text{ where } b_n = 2c_2c_3$$

Most general solution to problem 1 is

$$u_1(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \quad \ldots(4)$$

Now, $u_1(x, b) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a}$

where $b_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$

$\Rightarrow \quad b_n = \dfrac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx \quad \ldots(5)$

Similarly, the most general solution to problem 2 is

$$u_2\,(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right) \quad \ldots(6)$$

where $B_n = \dfrac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy \quad \ldots(7)$

| Interchanging x by y, a by b and $f(x)$ by $g(y)$

Hence, the required solution to the given original problem is

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

$$= \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) + B_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right)\right]$$

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where $$b_n = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

and $$B_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b g(y) \sin \frac{n\pi y}{b}\, dy.$$

EXERCISE E

1. A long rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$,

$$\lim_{y \to \infty} v(x, y) = 0 \quad \text{and} \quad v(x, 0) = kx$$

Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\left(\frac{n\pi y}{a}\right)} \sin \frac{n\pi x}{a}.$$

2. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 20) = x(20 - x)$ when $0 < x < 20$ while other three edges are kept at 0°C. Find the steady state temperature in the plate.

3. A rectangular plate has sides a and b. Let the side of length a be taken along OX and that of length b along OY and the other sides along $x = a$ and $y = b$. The sides $x = 0$, $x = a$ and $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \frac{\pi x}{a}$. Find the steady-state temperature at any point (x, y).

   [**Hint:** Boundary conditions are $(u_x)_{x=0} = 0$, $(u_x)_{x=a} = 0$, $(u_y)_{y=b} = 0$ and $u(x, 0) = u_0 \cos (\pi x/a)$]

4. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y).

   Hence, show that the temperature at the centre of the plate

$$= \frac{200}{\pi} \left[\frac{1}{\cosh \frac{\pi}{2}} - \frac{1}{3 \cosh \frac{3\pi}{2}} + \frac{1}{5 \cosh \frac{5\pi}{2}} - \ldots \right].$$

5. A rectangular plate is bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. Its surfaces are insulated and the temperature along the upper horizontal edge is 100°C while the other three edges are kept at 0°C. Find the steady state temperature function $u(x, y)$ and also the temperature at the point $\left(\frac{1}{2}a, \frac{1}{2}b\right)$.

6. Solve the following Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a rectangle with $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$ and $u(x, 0) = f(x)$ along x-axis.

7. Solve the boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 \le x \le a, \; 0 \le y \le b$$

   with the boundary conditions:

$$u_x(0, y) = u_x(a, y) = u_y(x, 0) = 0 \text{ and } u_y(x, b) = f(x)$$

8. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = 0$ and $u(x, 1) = 100 \sin \pi x$.

9. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the conditions $u(0, y) = 0$, $u(a, y) = 0$, $u(x, 0) = 0$ and $u(x, b) = x$.

10. Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to the conditions $u(x, 0) = 0$, $u(x, 1) = 0$, $u(\infty, y) = 0$ and $u(0, y) = u_0$.

11. The initial temperature distribution in a square plate of unit length is 100°C. Find the temperature distribution $u(x, t)$ if all the sides are maintained at zero degree temperature.

Answers

2. $$u(x, y) = \frac{3200}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi x}{20}\sin\frac{(2n-1)\pi y}{20}}{(2n-1)^3\sinh(2n-1)\pi}$$

3. $$u(x, y) = u_0\cos\frac{\pi x}{a}\cosh\frac{\pi}{a}(b-y)\operatorname{sech}\frac{\pi b}{a}$$

5. $$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left\{(2m-1)\frac{\pi x}{a}\right\}\sinh\left\{(2m-1)\frac{\pi y}{a}\right\}}{(2m-1)\sinh\left\{(2m-1)\frac{\pi b}{a}\right\}};$$

$$u\left(\frac{1}{2}a, \frac{1}{2}a\right) = \frac{200}{\pi}\left[\frac{1}{\cosh\left(\frac{\pi b}{2a}\right)}-\frac{1}{3\cosh\left(\frac{3\pi b}{2a}\right)}+\ldots\right]$$

6. $$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi}{a}(b-y),$$

where $$B_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

7. $$u(x, y) = \sum_{n=1}^{\infty} b_n\cos\frac{n\pi x}{a}\left(e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}\right) \text{ where, } b_n = \frac{1}{n\pi\cosh\frac{n\pi}{a}b}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

8. $$u(x, y) = 100\sin\pi x\left(\frac{\sinh\pi y}{\sinh\pi}\right)$$

9. $$u(x, y) = \frac{-2a}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}$$

10. $$u(x, y) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n}\right\}e^{-n\pi x}\sin n\pi y$$

11. $$u(x, y, t) = \frac{400}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1-\cos n\pi}{n}\right)\left(\frac{1-\cos m\pi}{m}\right)\sin n\pi x\sin m\pi y\, e^{-c^2k_{mn}^2 t}$$

where $k_{mn}^2 = \pi^2(n^2+m^2)$.

LAPLACE EQUATION

NOTES

Laplace's equation has wide applications in Physics and engineering. The theory of its solutions is called the *potential theory* and its solutions are called *harmonic functions*. The solution of Laplace's equation, subject to certain boundary conditions, is simplified by a proper choice of coordinate system.

Note 1. If the problem involves *rectangular boundaries*, we prefer to take Laplace's equation in *cartesian coordinates given by* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

Note 2. If the problem involves circular boundaries, we prefer to take Laplace's equation in polar coordinates given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$$

This equations can be obtained from $u_{xx}+u_{yy}=0$ by putting $x=r\cos\theta$, $y=r\sin\theta$, thus changing the independent variables from (x, y) to (r, θ).

Note 3. If the problem involves *cylindrical boundaries*, we prefer to take Laplace's equation in *cylindrical coordinates* given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

This equation can be obtained from $u_{xx}+u_{yy}+u_{zz}=0$ by putting $x=r\cos\theta$, $y=r\sin\theta$, $z=z$, thus changing the independent variables (x, y, z) to (r, θ, z).

Note 4. If the problem involves *spherical boundaries,* we prefer to take Laplace's equation in *spherical polar coordinates* given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}=0.$$

This equation can be obtained from $u_{xx}+u_{yy}+u_{zz}=0$ by putting,

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta,$$

thus changing the independent variables (x, y, z) to (r, θ, ϕ).

SOLUTIONS OF LAPLACE's EQUATION

(*a*) Solution of Laplace's Equation in Two-dimensional Cartesian Form

We have already discussed the solution of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ in Art. 2.16.

(*b*) Solution of Laplace's Equation in Polar Coordinates

Laplace's equation in polar coordinates is $r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0$ ...(1)

Let $u(r, \theta)=R(r)F(\theta)$ or simply $u = RF$ ...(2)

where R is a function of r only and F is a function of θ only, be a solution of (1).

Substituting it in (1), we get

$$r^2R''F+rR'F+RF''=0 \quad \text{or} \quad (r^2R''+rR')F+RF''=0$$

Separating the variables, $\frac{r^2R''+rR'}{R}=-\frac{F''}{F}=\text{constant}=k$ (say)

Thus, we get ordinary differential equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \quad ...(3)$$

and $$\frac{d^2F}{d\theta^2}+kF=0 \quad ...(4)$$

Now (3) is a homogeneous linear differential equation.

Putting $r = e^z$, (3) reduces to $\frac{d^2R}{dr^2}-kR=0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive and* $=p^2$, *say*

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},$$
$$F = c_3 \cos p\theta + c_4 \sin p\theta.$$

(*ii*) *When k is negative and* $= -p^2$, *say*

$$R = c_1 \cos pz + c_2 \sin pz = c_1 \cos (p \log r) + c_2 \sin (p \log r)$$
$$F = c_3e^{p\theta} + c_4e^{-p\theta}$$

(*iii*) *When k = 0* $\quad R = c_1z + c_2 = c_1 \log r + c_2$

$$F = c_3\theta + c_4$$

Thus the three possible solution of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(6)$$
$$u = [c_1 \cos (p \log r) + c_2 \sin (p \log r)]\ (c_3e^{p\theta} + c_4e^{-p\theta}) \quad ...(7)$$
$$u = (c_1 \log r + c_2)(c_3\theta + c_4) \quad ...(8)$$

Of these solutions, we choose the one which is consistent with the physical nature of the problem.

Note. Usually we require a solution extending up to the origin.

Since u must be finite at the origin, we reject solutions (7) and (8). Also from (6), $c_2 = 0$.

$\therefore$ In this case, the solution may be written as

$$u = (A \cos p\theta + B \sin p\theta)r^p$$

The general solution will consist of a sum of similar terms with different (arbitrary) values of A, B and p.

(*c*) Solution of Laplace's Equation in Three-dimensional Cartesian Form

Laplace's equation in three-dimensional cartesian form is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad ...(1)$$

Let $u(x, y, z) = X(x)\,Y(y)\,Z(z)$ or simply $u = XYZ$ ...(2) be a solution of (1).

Substituting it in (1), we get $X''YZ + XY''Z + XYZ'' = 0$

Dividing to XYZ, we get $\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=0$

or $$\frac{1}{X}\cdot\frac{d^2X}{dx^2}+\frac{1}{Y}\cdot\frac{d^2Y}{dy^2}+\frac{1}{Z}\cdot\frac{d^2Z}{dz^2}=0 \quad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

Since *x, y, z* are independent, this is possible only when F_1, F_2, F_3 are constants. Assuming these constants to be k^2, l^2 and $-(k^2 + l^2)$ respectively, (3) gives rise to the following equations:

$$\frac{d^2X}{dx^2}-k^2X=0,\ \frac{d^2Y}{dy^2}-l^2Y=0,\ \frac{d^2Z}{dz^2}+(k^2+l^2)Z=0$$

Their solutions are $\quad X = c_1e^{kx} + c_2e^{-kx},\ Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2+l^2)}\,z + c_6 \sin \sqrt{(k^2+l^2)}\,z$$

Hence a solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})[c_5 \cos \sqrt{(k^2 + l^2)}\, z + c_6 \sin \sqrt{(k^2 + l^2)}\, z]$$

Since, the three constants could have been taken as $-k^2$, $-l^2$ and $k^2 + l^2$, and alternative solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\, [c_5 e^{\sqrt{(k^2+l^2)}\, z} + c_6 e^{-\sqrt{(k^2+l^2)}\, z}]$$

The choice of the constants and hence the general solution depends on the given initial and boundary conditions.

(*d*) **Solution of Laplace's Equation in Cylindrical Coordinates**

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

Let $u(r, \theta, z) = \mathrm{R}(r)\mathrm{F}(\theta)\mathrm{Z}(z)$

or simply $u = \mathrm{RFZ}$ ...(2) be a solution of (1).

Substituting it in (1), we get $\mathrm{R''FZ} + \frac{1}{r}\mathrm{R'FZ} + \frac{1}{r^2}\mathrm{RF''Z} + \mathrm{RFZ''} = 0$

Dividing by RFZ, we get

$$\frac{1}{\mathrm{R}}\left(\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr}\right) + \frac{1}{r^2\mathrm{F}}\frac{d^2\mathrm{F}}{d\theta^2} + \frac{1}{z}\frac{d^2\mathrm{Z}}{dz^2} = 0 \qquad ...(3)$$

Assuming $\dfrac{d^2\mathrm{F}}{d\theta^2} = -n^2\mathrm{F}$ and $\dfrac{d^2\mathrm{Z}}{dz^2} = k^2\mathrm{Z}$ ...(4)

Equation (3) reduces to $\dfrac{1}{\mathrm{R}}\left(\dfrac{d^2\mathrm{R}}{dr^2} + \dfrac{1}{r}\dfrac{d\mathrm{R}}{dz}\right) - \dfrac{n^2}{r^2} + k^2 = 0$

or $$r^2\frac{d^2\mathrm{R}}{dr^2} + r\frac{d\mathrm{R}}{dz} + (k^2r^2 - n^2)\mathrm{R} = 0$$

This is **Bessel's equation.** Its solution is $\mathrm{R} = c_1\mathrm{J}_n(kr) + c_2\mathrm{Y}_n(kr)$

The solutions of equations (3) are $\mathrm{F} = c_3 \cos n\theta + c_4 \sin n\theta$, $\mathrm{Z} = c_5e^{kz} + c_6e^{-kz}$

Hence a solution of (1) is $u = [c_1\mathrm{J}_n(kr) + c_2\mathrm{Y}_n(kr)]\,(c_3 \cos n\theta + c_4 \sin n\theta)(c_5e^{kz} + c_6e^{-kz})$ which is known as a *cylindrical harmonic.*

(*e*) **Solution of Laplace's Equation in Spherical Coordinates**

Laplace's equation is spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad ...(1)$$

Let $u(r, \theta, \phi) = \mathrm{R}(r)\mathrm{G}(\theta)\mathrm{H}(\phi)$.

or simply $u = \mathrm{RGH}$ ...(2) be a solution of (1).

Substituting it in (1), we get

$$\mathrm{R''GH} + \frac{2}{r}\mathrm{R'GH} + \frac{1}{r^2}\mathrm{RG''H} + \frac{\cot\theta}{r^2}\mathrm{RG'H} + \frac{1}{r^2\sin^2\theta}\mathrm{RGH''} = 0$$

Dividing by RGH, we get

$$\frac{1}{\mathrm{R}}\left(r^2\frac{d^2\mathrm{R}}{dr^2} + 2r\frac{d\mathrm{R}}{dr}\right) + \frac{1}{\mathrm{G}}\left(\frac{d^2\mathrm{G}}{d\theta^2} + \cot\theta\frac{d\mathrm{G}}{d\theta}\right) + \frac{1}{\mathrm{H}\sin^2\theta}\frac{d^2\mathrm{H}}{d\phi^2} = 0 \qquad ...(3)$$

NOTES

Putting $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \quad ...(4)$$

and $$\frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2 \quad ...(5)$$

Equation (3) reduces to $$\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]\,G=0$$

This is **associated Legendre's equation** and its solution is

$$G=c_1P_n^m(\cos\theta)+c_2\,Q_n^m(\cos\theta)$$

The solution of (5) is $H=c_3\cos m\phi+c_4\sin m\phi$

To solve (4), assume that $R=r^k$ so that

$$k(k-1)+2k=n(n+1) \quad \text{or} \quad (k^2-n^2)+(k-n)=0$$

or $$(k-n)(k+n+1)=0 \quad \therefore \quad k=n \text{ or } -n-1$$

Thus $$R=c_5r^n+c_6r^{-n-1}$$

Hence, the general solution of (1) is

$$u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)](c_3\cos m\phi+c_4\sin m\phi)(c_5r^n+c_6r^{-n-1})$$

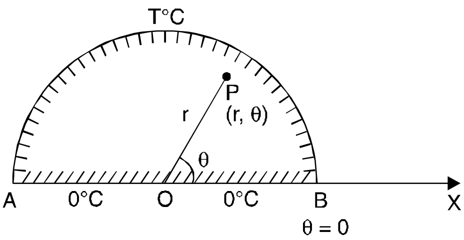
Any solution of (1) is known as a *spherical harmonic.*

SOLVED EXAMPLES

Example 39. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$$

Sol. Take the centre of the circle as the pole and the bounding diameter as the initial line. Let the steady state temperature at any point P(r, θ) be $u(r,\theta)$, so that u satisfies the equation



$$r^2\frac{\partial^2u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2u}{\partial\theta^2}=0 \quad ...(1)$$

Let $$u=RT \quad ...(2)$$

where R is a function of r only and T is a function of θ only.

$$\frac{\partial u}{\partial r}=\frac{\partial}{\partial r}(RT)=T\frac{dR}{dr}$$

$$\frac{\partial^2u}{\partial r^2}=\frac{\partial}{\partial r}\left(T\frac{dR}{dr}\right)=T\frac{d^2R}{dr^2}$$

$$\frac{\partial^2u}{\partial\theta^2}=\frac{\partial^2}{\partial\theta^2}(RT)=R\frac{d^2T}{d\theta^2}.$$

NOTES

$\therefore$ From (1), $r^2TR'' + rTR' + RT'' = 0$

$$\frac{r^2R'' + rR'}{R} + \frac{T''}{T} = 0 \Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{T''}{T}$$

Case I. *When* $\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = p^2$ *(say)*

(*i*) $\frac{r^2R'' + rR'}{R} = p^2 \Rightarrow r^2R'' + rR' - p^2R = 0.$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$ then above equation reduces to

$$\{D(D-1) + D - p^2\} R = 0$$

$$(D^2 - p^2)R = 0$$

Auxiliary equation is $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$\text{C.F.} = c_1e^{pz} + c_2e^{-pz}$$

$$= c_1e^{p \log r} + c_2e^{-p \log r} = c_1r^p + c_2r^{-p}$$

$$\text{P.I.} = 0.$$

$\therefore$ $R = c_1r^p + c_2r^{-p}$

(*ii*) $-\frac{T''}{T} = p^2$

$$\frac{d^2T}{d\theta^2} + p^2T = 0.$$

Auxiliary equation is $m^2 + p^2 = 0$

$\Rightarrow$ $m = \pm pi$

$\therefore$ $\text{C.F.} = c_3 \cos p\theta + c_4 \sin p\theta$

$$\text{P.I.} = 0$$

$\therefore$ $T = c_3 \cos p\theta + c_4 \sin p\theta$

$\therefore$ $u(r, \theta) = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$ ...(3)

Case II. *When* $\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = -p^2$ *(say)*

(*i*) $\frac{r^2R'' + rR'}{R} = -p^2$

$\Rightarrow$ $r^2R'' + rR' + p^2R = 0$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \frac{d}{dz}$ then above equation reduces to

$$[D(D-1) + D + p^2] R = 0$$

$$(D^2 + p^2)R = 0.$$

Auxiliary equation is $m^2 + p^2 = 0$

$$m = \pm pi$$

$$\text{C.F.} = (c_5 \cos pz + c_6 \sin pz)$$

$$\text{P.I.} = 0$$

$\therefore$ $R = c_5 \cos pz + c_6 \sin pz$

$$= c_5 \cos (p \log r) + c_6 \sin (p \log r).$$

NOTES

(*ii*) $-\dfrac{T''}{T} = -p^2$

$\Rightarrow$ $\dfrac{T''}{T} = p^2$ or $\dfrac{d^2T}{d\theta^2} - p^2T = 0.$

Auxiliary equation is $m^2 - p^2 = 0$

$m = \pm p$

$\therefore$ C.F. $= c_7e^{p\theta} + c_8e^{-p\theta}$

P.I. $= 0.$

$\therefore$ $T = c_7e^{p\theta} + c_8e^{-p\theta}.$

Hence, $u(r, \theta) = [c_5 \cos(p \log r) + c_6 \sin(p \log r)]$

$(c_7e^{p\theta} + c_8e^{-p\theta})$ ...(4)

Case III. *When* $\dfrac{r^2R'' + rR'}{R} = -\dfrac{T''}{T} = 0$ *(say)*

(*i*) $r^2R'' + rR' = 0.$

Put $r = e^z$ so that $z = \log r$ and let $D \equiv \dfrac{d}{dz}$, then above equation reduces to

$[D(D-1) + D]R = 0$

$D^2R = 0$

Auxiliary equation is $m^2 = 0$

$\Rightarrow$ $m = 0, 0$

$\therefore$ C.F. $= (c_9 + c_{10}z)\, e^{0z} = c_9 + c_{10} \log r$

P.I. $= 0$

$\therefore$ $R = c_9 + c_{10} \log r.$

(*ii*) $-\dfrac{T''}{T} = 0$

$\Rightarrow$ $T'' = 0$

$\Rightarrow$ $T = c_{11} + c_{12}\theta.$

$\therefore$ $u(r, \theta) = (c_9 + c_{10}\log r)(c_{11} + c_{12}\theta)$ ...(5)

Of these three solutions (3), (4) and (5), we choose the solution consistent with the given boundary conditions.

Boundary conditions are

$u(r, 0) = 0$ ...(6)

$u(r, \pi) = 0$ ...(7)

$u(a, \theta) = T$ ...(8)

and $u \to 0$ as $r \to 0$ ...(9)

Solutions (4) and (5) do not satisfy boundary condition (9).

Hence, the consistent solution is

$u(r, \theta) = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$ ...(10)

From (10), $u(r, 0) = 0 = (c_1r^p + c_2r^{-p})\, c_3$

$\Rightarrow$ $c_3 = 0.$

From (10),

$u(r, \theta) = (c_1r^p + c_2r^{-p})\, c_4 \sin p\theta$ ...(11)

$u(r, \pi) = 0 = (c_1r^p + c_2r^{-p})\, c_4 \sin p\pi$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi \ (n \in \text{I})$

$\therefore$ $p = n.$

$\therefore$ From (11), $u(r, \theta) = (c_1 r^n + c_1 r^{-n})\, c_4 \sin n\theta$ ...(12)

Condition $u \to 0$ as $r \to 0$ is satisfied only and only when $c_2 = 0$.

Hence from (12), $u(r, \theta) = c_1 c_4 r^n \sin n\theta = b_n r^n \sin n\theta$.

The most general solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad \text{...(13)}$$

$$u(a, \theta) = \text{T} = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

where $$b_n a^n = \frac{2}{\pi}\int_0^{\pi} \{\text{T} \sin n\theta \, d\theta\}$$

$$= \frac{2\text{T}}{\pi}\left(\frac{-\cos n\theta}{n}\right)_0^{\pi} = \frac{2\text{T}}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2\text{T}}{n\pi}\{1 - (-1)^n\} = \begin{cases} \dfrac{4\text{T}}{n\pi}; & n \text{ is odd} \\ 0\ ; & n \text{ is even} \end{cases}$$

$\therefore$ $$b_n = \begin{cases} \dfrac{4\text{T}}{n\pi a^n}; & n \text{ is odd} \\ 0\ ; & n \text{ is even} \end{cases}$$

From (13), $$u(r, \theta) = \frac{4\text{T}}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot \frac{r^n}{a^n} \sin n\theta$$

$$u(r, \theta) = \frac{4\text{T}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\theta$$

which is the required steady state temperature in the plate.

Example 40. *Solve:* $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = 0$ *with boundary conditions*

(*i*) *V is finite when* $r \to 0$ (*ii*) $V = \Sigma\, C_n \cos n\theta$ *on* $r = a$.

Sol. Solution to given differential equation is

$$\text{V} = \Sigma\,(\text{A}_n r^n + \text{B}_n r^{-n}) \cos (n\theta + \alpha)$$

When $r = a$, $\text{V} = \Sigma\, \text{C}_n \cos n\theta$

$\therefore$ $\Sigma\, \text{C}_n \cos n\theta = \Sigma(\text{A}_n a^n + \text{B}_n a^{-n}) \cos (n\theta + \alpha)$

$\therefore$ $\text{C}_n = \text{A}_n a^n + \text{B}_n a^{-n},\ \alpha = 0$

When $r \to 0$, V is finite.

$\therefore$ $\text{B}_n = 0$ (otherwise V becomes ∞)

$\therefore$ $$\text{A}_n = \frac{\text{C}_n}{a^n}$$

$\therefore$ $$\text{V} = \Sigma\, \text{C}_n \left(\frac{r}{a}\right)^n \cos n\theta.$$

Example 41. *The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.*

Sol. Here, we have to take the solution in polar coordinates.

The solution is

$$u = (c_1 \cos p\theta + c_2 \sin p\theta)(c_3 r^p + c_4 r^{-p}) \quad ...(1)$$

Since, the temperature remains finite at $r = 0$

$$\therefore \quad c_4 = 0 \quad ...(2)$$

Also, if we increase θ by 2π, we arrive at the same point. So the solution (1) should be periodic with period 2π.

Therefore $p = n$, an integer. Hence, we may write the general solution as

$$u = \sum_{n=0}^{\infty} (c_1 \cos n\theta + c_2 \sin n\theta) c_3 r^n$$

$$= \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) r^n \quad | \; c_1c_3 = A_n \; c_2c_3 = B_n \text{ (say)}$$

Applying to this, the condition

$$u = f(\theta) \quad \text{for} \quad r = a, \text{ we get}$$

$$f(\theta) = \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) a^n$$

where $$a^n A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

and $$a^n B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

Example 42. *Find a harmonic function ϕ in the semi-circle $r < a$, $0 < \theta < \pi$ which vanishes on $\theta = 0$ and takes the value c on $\theta = \pi$ and on the curved portion $r = a$.*

Sol. A harmonic function is a function satisfying Laplace's equation.

Solution to $r^2 \dfrac{\partial^2 \phi}{\partial r^2} + r \dfrac{\partial \phi}{\partial r} + \dfrac{\partial^2 \phi}{\partial \theta^2} = 0$ is

$$\phi = (A \cos p\theta + B \sin p\theta)(Cr^p + Dr^{-p}) \quad ...(1)$$

Since $\phi \to 0$ as $r \to 0 \quad \therefore \quad D = 0$

$$\therefore \quad \phi = (A \cos p\theta + B \sin p\theta) Cr^p \quad ...(2)$$

Imposing on (2), the zero boundary conditions

$$\phi(r, 0) = 0 \quad \text{and} \quad \phi(r, \pi) = 0, \text{ we get}$$

$$A = 0 \quad \text{and} \quad p = n \; (n \in I)$$

Putting in (2) and adding up various solutions for $n = 1, 2, 3, ...,$ we get

$$\phi = \sum_{n=1}^{\infty} B_n r^n \sin n\theta \quad ...(3)$$

NOTES

This solution will not satisfy both the given boundary conditions of the problem namely

$$\left.\begin{aligned} &(i)\ \phi = 0 \text{ when } \theta = 0 \\ \text{and}\quad &(ii)\ \phi = c \text{ when } \theta = \pi \end{aligned}\right\} \qquad \text{...(4)}$$

So we add to (3), the solution $\phi = A_0\theta + B_0$ which satisfies Laplace's equation. We choose A_0, B_0 so that (4) is satisfied.

Then, $B_0 = 0$ and $c = A_0\pi$

Hence, $$\phi = \frac{c\theta}{\pi} + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \qquad \text{...(5) satisfies (4)}$$

Applying to (5), the condition that $\phi = c$ when $r = a$, we get

$$c = \frac{c\theta}{\pi} + \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$\Rightarrow \quad c\left(1 - \frac{\theta}{\pi}\right) = \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

$$\therefore \quad B_n a^n = \frac{2}{\pi}\int_0^{\pi} c\left(1 - \frac{\theta}{\pi}\right)\sin n\theta\, d\theta$$

$$B_n a^n = \frac{2c}{n\pi} \quad \Rightarrow \quad B_n = \frac{2c}{n\pi a^n} \qquad \text{...(6)}$$

$\therefore$ From (5) and (6),

$$\phi = \frac{c\theta}{\pi} + \frac{2c}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta$$

EXERCISE F

1. Show that the steady state temperature distribution in a semi-circular plate of radius a whose bounding diameter is kept at 0°C, while the circumference is kept at 60°C is given by

$$u(r, \theta) = \frac{240}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1} . \sin (2n-1)\theta.$$

2. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the bounding diameter is kept at zero temperature. Assuming the surfaces of the plate to be insulated, show that the steady-state temperature distribution of the plate is given by

$$u(r, \theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{2n-1} . \frac{\sin (2n-1)\theta}{(2n-1)^3}.$$

3. The bounding diameter of a semi-circular plate of radius a is kept at 0°C and the temperature along the semi-circular boundary is given by

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta < \dfrac{\pi}{2} \\ 50(\pi - \theta), & \text{when } \dfrac{\pi}{2} < \theta < \pi \end{cases}.$$

Show that the steady-state temperature distribution given by

$$u(r, \theta) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}\left(\frac{r}{a}\right)^{2n-1} . \sin (2n-1)\theta.$$